This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

QA275
D6
1844

STANFORD
LIBRARIES

AN ESSAY
ON THE THEORY OF THE COMBINATION OF OBSERVATIONS.

BY

W. F. DONKIN, M.A. F.R.S.

OF UNIVERSITY COLLEGE,

SAVILIAN PROFESSOR OF ASTRONOMY

OXFORD:

PRINTED BY THOMAS COMBE, PRINTER TO THE UNIVERSITY, FOR

THE ASHMOLEAN SOCIETY.

QA275
D6
1844

STANFO
LIBRARI

AN ESSAY

ON THE THEORY OF THE COMBINATION OF OBSERVATIONS.

BY

W. F. DONKIN, M.A. F.R.S.

OF UNIVERSITY COLLEGE,

SAVILIAN PROFESSOR OF ASTRONOMY.

OXFORD:

PRINTED BY THOMAS COMBE, PRINTER TO THE UNIVERSITY, FOR

THE ASHMOLEAN SOCIETY.

M.DCCC.XLIV.

STANFORD UNIVERSITY LIBRARIES

AN ESSAY

ON THE THEORY OF THE

COMBINATION OF OBSERVATIONS.

BY

W. F. DONKIN, M.A. F.R.S.

OF UNIVERSITY COLLEGE,

SAVILIAN PROFESSOR OF ASTRONOMY.

OXFORD:

PRINTED BY THOMAS COMBE, PRINTER TO THE UNIVERSITY, FOR

THE ASHMOLEAN SOCIETY.

M. DCCC. XLIV.

QA275
D6
1844

The Ashmolean Society desire it to be understood that they are not answerable, as a body, for any facts, reasonings, or opinions, advanced in papers printed by them.

ON THE

THEORY OF THE COMBINATION OF OBSERVATIONS.

PRESENTED TO THE ASHMOLEAN SOCIETY, FEB. 26, 1844.

IT is well known that the method of Least Squares does not, strictly speaking, give the most probable result to be drawn from a system of observations, unless either the number of observations be indefinitely great, or the function expressing the relative facility of errors have a particular form; whilst it is certain that, in practice, neither of these conditions ever subsists otherwise than approximately.

Yet it seems to be admitted that the method in question does strictly, in some sense or other, give the *best* result which can be obtained. No one, however, attempted a demonstration of this principle before Professor Gauss. In his treatise entitled "Theoria Combinationis Observationum erroribus minimis obnoxiae," (Göttingen, 1823), he has considered the following question: Admitting that the number of observations is finite, and that we do not know the form of the function by which their character is defined, what conclusion ought we to adopt under these circumstances? or, to employ his own term, what is the most *plausible* result? And he has demonstrated that the method of Least Squares must be used, provided we assume that result to be most plausible which would give the least average value to the *square of the error* in an infinite number of trials. (*Theor. Comb.* art. 6. and 21.) But this assumption, however strongly it may commend itself to the instinct of the mathematician, confessedly involves something arbitrary, and so there is still something wanting to the logical consistence of the theory which rests upon it.

In the following essay the subject is considered under a different aspect, but the results are identical with those obtained in the treatise just referred to, at least so far as they are comparable with them. The reasoning employed professes to be demonstrative in the same sense in which the ordinary *a priori* proofs of the fundamental theorems of Statics profess to be so; and as it is important that it should be clearly understood in what sense this is, I have added some remarks at the end upon this point. But whatever may be thought of the logical character of the processes, I believe it will be allowed that they contain nothing *arbitrary;* that no principle is assumed which is not obviously and unquestionably the most plausible, or rather the only plausible one which could be assumed at all. And if so, the complete coincidence of the conclusions with those deduced from a perfectly distinct and independent set of principles may be looked upon as an interesting fact, notwithstanding the failure of any particular attempt to explain it.

SECTION I.

1. If the value of an unknown quantity x has been determined by one observation not absolutely accurate to be $=x_1$, and by another observation to be $=x_2$, and if there are no other data, it is plain that we must assume for it some value x, intermediate between x_1 and x_2. Suppose x (algebraically) less than x_2. Then the first observation, considered by itself, supplies a reason for *diminishing x;* the second, a reason for *increasing* x. Therefore x must be so assumed that these two reasons may exactly counterbalance one another. Let *force* be defined to be *reason for alteration*, and let a quantity be said to be in equilibrio when the force tending to diminish it equals the force tending to increase it. We have now to shew how such forces can be numerically compared, and to investigate the laws of their equilibrium.

2. Let two observations be said to be of the same kind, when two numbers m and n can be found, such that if m repetitions of the first observation concurred in giving the same value for the unknown quantity, our knowledge of it would be exactly the same as if n repetitions of the second had similarly concurred. Let the reciprocals of m and n be taken to measure the *weights* or *absolute forces*[a] of the respective observations. Thus

---

[a] It is important to observe that this is a mere definition, and affirms no proposition whatever.

if we take a particular observation as a standard, and call its weight 1, the weight of any other observation will be represented by g, if $g=\frac{m}{n}$, and n repetitions of it be equivalent to m repetitions of the standard observation. In what follows, all the observations are supposed to be of the same kind, i. e. comparable in respect of their weights.

3. The determination of a single unknown quantity x by direct observation, may be considered as the determination of the position of a point in a straight line by means of its distance from a fixed point in the same line.

Suppose two observations of the same weight g give two different positions A and B for the unknown point P. It is manifest that we must place it at a point C half way between A and B, since there is no reason for placing it nearer to one of these points than to the other.

4. The *weight* of this determination is $2g$. That is, our knowledge of the position of P is exactly the same as if one observation of weight $2g$ had placed it at C.

For let a be the distance between A and B. Then if the weight in question be not equal to $2g$, let it $=\gamma$, and γ must depend in some way upon g and a. Suppose

$$\gamma=g\cdot\phi(a)$$

(the equation must be homogeneous with respect to γ and g, otherwise its meaning would be altered by altering the unit of weight). We know that $\phi(o)=2$, since if both observations had concurred, the weight of the result would have been $2g$ by definition. Also $\phi(a)$ cannot be a function varying continuously with a; for if it were, the meaning of the equation would be altered by altering the unit of linear measure. Therefore either $\phi(a)$ is constant and $=2$, or it is discontinuous, and $=2$ when $a=0$, but has some other constant value when a is different from nothing.

I shall assume that the supposition of discontinuity is inadmissible[b], and consequently that $\gamma=2g$.

5. Hence we may substitute for the results of two observations of equal weight g, their arithmetical mean; and consider it to be the result of a single observation of weight $2g$. And conversely, for a single observation of weight g, giving $x=x_0$, we may substitute two observations of weight $\frac{1}{2}g$, giving $x=x_0+c$, $x=x_0-c$ respectively, where c is entirely arbitrary.

Consequently, by a process identical with the well-known Archimedean demonstration of the equilibrium of the lever, it follows that if a number of observations whose weights are $g_1, g_2, g_3 \ldots\ldots$ give respectively $x_1, x_2, x_3, \ldots\ldots$ for the value of an unknown quantity x, we may substitute for them a single imaginary observation giving

$$x=\frac{g_1 x_1+g_2 x_2+g_3 x_3+\ldots}{g_1+g_2+g_3+\quad\ldots} \tag{1}$$

with weight $=g_1+g_2+g_3+\ldots\ldots$

---

[b] The same assumption is necessary in the theory of parallel forces. The expression for the resultant of two equal and parallel forces cannot involve a continuous function of the distance between them; but it might conceivably be equal to their sum when the distance was nothing, and have some constant ratio (not unity) to their sum when the distance was not nothing.

Let the value of x thus determined be called the *most plausible value.*

6. It is now easy to discover the law of variation of the force arising from any observation.

Let g be the weight of an observation which assigns A for the position of an unknown point in a given straight line. If we place the point in any other position P, at a distance r from A, the variation of the force f urging it towards A must be expressed by a homogeneous equation of the form

$$\frac{f}{f_1} = \phi\left(\frac{r}{r_1}\right)$$

where f_1 is the force at any other distance r_1. Also, the distance remaining the same, the force is proportional to the weight of the observation. For suppose m concurrent observations, of weight g, had assigned A for the position of the point. Then if we placed it at a distance r_1 from A, m forces each equal to f_1 would act upon it. But m observations of weight g are equivalent to one observation of weight mg; whence the truth of the proposition is manifest; and since, r_1 being given, f_1 is proportional to g, g may be taken as the measure of f_1.

If therefore, for greater simplicity, we assume the unit of distance for the arbitrary r_1, the equation above written becomes

$$f = g . \phi(r)$$

and it only remains to determine the form of the function ϕ.

7. Equation (1) art. 5, shews that the position of equilibrium of the point P is the point which

would be the Centre of Gravity of the points assigned by the separate observations, supposing a mass to be placed at each of them proportional to the weight of the corresponding observation. The question therefore obviously reduces itself to finding the law of attraction which would give the Centre of Gravity of any number of material points placed in a straight line for the position of rest of a point attracted by each of them. Now it is well known that when the force varies directly as the distance, this condition is satisfied whatever be the arrangement of the attracting masses, since the attraction of any material system is, on this supposition, the same as if it were concentrated at its Centre of Gravity. Also there is no other law which satisfies the same condition. For let there be only two points; μ, μ' their masses; r, r' their distances from the Centre of Gravity. Then if $\phi(r)$ express the law of force, and if the attraction on a point at the Centre of Gravity be $= o$ we must have

$$\mu\,\phi(r) = \mu'\,\phi(r')$$

but

$$\mu r = \mu' r'$$

and therefore

$$\frac{\phi(r)}{r} = \frac{\phi(r')}{r'}$$

or $\frac{\phi(r)}{r}$ is constant, and the force varies directly as the distance.

8. We have thus established the following theorem.

If any value x be assumed for an unknown quantity which has been given $= x_\circ$ by an observation of weight g, then the force arising from that observation and tending to alter x is proportional to, and may be expressed by, the product $g.\,(x - x_\circ)$.

In order to avoid the frequent use of negative signs, it will be convenient to consider those forces positive which tend to *diminish* (algebraically) the quantities on which they act.

9. The theorem in the last article was obtained by the help of a construction in which the unknown quantity was made to represent the distance of a point in a straight line from a given point in the same. But we might equally well have considered it to be the distance of a point in a plane from a given straight line in the same plane; or the distance of a point in space from a given plane. In this last case, for example, each observation would assign for the locus of the point a certain plane parallel to the given plane; and in any assumed position, the force arising from each observation would be measured by the product of its weight into the distance of the point from the corresponding plane. Whichever construction be adopted, the theorem itself is perfectly general.

10. Hitherto we have considered only the case in which the directions of the forces acting upon a point are all in the same straight line. But if the position of the point were determined by observations assigning its distance from straight lines or planes not parallel to one another, the forces acting on it in its position of equilibrium would have different directions. Now the fundamental conditions to which the forces we are considering are subject, are exactly the same as those which form the foundation of the ordinary investigations of Statics. They are

(1) That the forces are capable of *addition;* or that any two forces acting at once in the same

direction upon a point, may be counterbalanced by a single force acting upon it in the contrary direction.

(2) That a force produces no tendency to (or reason for) motion in a direction perpendicular to itself.

These two conditions, together with the principle of homogeneity, suffice for the demonstration of the parallelogram of forces, and consequently of the whole theory of the equilibrium of a single point. (See Appendix A.)

11. And from these conclusions, the conditions of equilibrium of any system of points whatever, and in particular, the principle of virtual velocities, which is the general expression of them all, are deducible by the help of the two ideas of a perfectly *rigid* and of a perfectly *flexible* line, with their fundamental properties; which properties are exactly the same with respect to forces of the kind here considered, as with respect to those with which the ordinary science of Statics is concerned.

12. The whole theory of equilibrium may therefore be stated as follows:

If v, v', v'', ... be a system of values attributed to quantities connected by any conditions whatever; and if f, f', f''.... be the forces tending respectively to alter their values, the condition of equilibrium is

$$fdv + f'dv' + f''dv'' + \dots \quad = 0.$$ [c]

---

[c] For whatever be the number of the quantities v, v', v'', ... and the nature of the relations between them, they may be considered as representing the distances of a system of points in a straight line, connected by corresponding conditions, from a fixed point in the same, and then the principle of virtual velocities is immediately applicable.

This theorem, together with that stated in Art. 8, will form the basis of all that follows.

SECTION II.

13. If, instead of determining the value of x by direct observation, we concluded it from an observation of any multiple, nx, of x, the error in the value so concluded would be only $\frac{1}{n}$th of the error of the observation. In other words, the *precision* with which x is determined is n times as great as that with which nx is determined.

If the value of x, determined directly by an observation A, be supposed to be thereby given with exactly the same certainty as the value concluded from another observation B of nx, then the precision of the observation A is said to be n times as great as that of B; and if we call the precision of B unity, that of A will be represented by n. And similarly, if an observation A of mx and another B of nx give values of x equally entitled to confidence, the precision of A is to that of B as $n : m$.

14. It is important to determine the relation between *precision* and *weight*; and this may be done immediately by means of the principles established in the preceding section.

Let $y = mx$. And suppose an observation whose precision is unity and weight g has given $y = y_0$. The deduced value of x, i.e. $\frac{y_0}{m}$ or x_0, will be entitled to the same confidence as if it had been determined by a direct observation, of precision m. Now if we assume for x and y any corresponding values different from x_0 and y_0, the force tending to alter y is $g(y-y_0)$ (Art. 8); let f be the force tending to alter x; then if the latter force acted in a contrary sense, the system would be in equilibrio. Therefore by the principle of virtual velocities (Art. 12),

$$fdx = g\,(y-y_0)\,dy,$$

or, since $y-y_0 = m\,(x-x_0)$ and $dy = mdx$,

$$f = m^2g.\,(x-x_0)$$

i. e. the force acting on x is the same as if this quantity had been given $=x_0$ by a direct observation, of *weight* m^2g. But we have seen that x may be considered to have been determined by a direct observation, of *precision* m. Consequently we have the following Theorem:

If g be the weight of an observation whose precision is unity, then m^2g is the weight of an observation whose precision is m. Or, the *weight* varies as the *square of the precision* [d].

15. It follows immediately from the definition of precision, that the risk of any error e in the result of an observation whose precision is m, is the same as the risk of an error me in an observation whose precision is unity.

And therefore the theorem proved in the last article shews that if e^2, e'^2, e''^2, ... be the squares of errors equally to be feared in the results of observations whose weights are respectively $g, g', g'' \ldots.$ then $\quad ge^2 = g'e'^2 = g''e''^2 = \ldots.$

---

[d] The *assumption* of an equivalent principle in the *Theoria Combinationis*, Art. 6, as the foundation of the whole theory, constitutes the doubtful step alluded to in the introduction to this essay.

SECTION III.

16. If u be a given function of any number of independent quantities $v, v', v'' \ldots.$ suppose

$$u = F(v, v', v'', \ldots.)$$

and if the values of $v, v', v'', \ldots$ have been given respectively equal to $v_0, v_0', v_0'', \ldots.$ by observations whose weights are $g, g', g'' \ldots..$ then, so long as we know nothing more, $v_0, v'_0, \ldots$ are the most plausible values of $v, v', v'', \ldots.$ and $u_0 = F(v_0, v_0', v_0'' \ldots.)$ is the most plausible value of u.

Suppose now that we discover *directly* that u has really some value u_1, different from u_0. Then $v_0, v'_0, \ldots$ are no longer the most plausible values of $v, v', \ldots..$ But it is evident that out of all systems of values of these quantities which give $u = u_1$, that system must be chosen which is *in equilibrio* under the forces arising from the original observations.

Now the condition of equilibrium is (Arts. 8, 12)

$$g(v - v_0)\,dv + g'(v' - v'_0)\,dv' + \ldots \quad = 0$$

and since the variations of $v, v', \ldots$ are subject to the condition $u = u_1$, we have also

$$\frac{du}{dv}\,dv + \frac{du}{dv'}\,dv' + \ldots \quad = 0$$

and there is no other relation between the differentials. Consequently

$$\frac{g(v-v_0)}{\frac{du}{dv}} = \frac{g'(v'-v'_0)}{\frac{du}{dv'}} = \frac{g''(v''-v''_0)}{\frac{du}{dv''}} = \ldots . \quad (1)$$

This formula, combined with the equation $u = u_1$, gives as many equations as there are quantities $v, v', \ldots$ to be determined, and thus we obtain the most plausible values of these quantities consistent with the supposition $u = u_1$.

17. The formula (1) of the last Article, considered by itself, expresses the law according to which $v, v', v'', \ldots .$ must vary, when *different* suppositions are made with respect to the value of u. I shall call this law the law of *relative equilibrium.* If we call f the force acting on u (when any value different from u_0 is given to it), then, since if it acted in the contrary sense, the system would be in equilibrio, we must have

$$fdu = g(v-v_0)\,dv + g'(v'-v'_0)\,dv' + \ldots .$$

where $$du = \frac{du}{dv}dv + \frac{du}{dv'}dv' + \ldots .$$

and $v, v', v'', \ldots . dv, dv', \ldots .$ are subject to all the relations implied by the formula (1). These principles might be presented in a more general form, but enough has been said to make the extension of them easy, if any particular case should require it[e].

---

[e] This subject may be illustrated by an analogous case in mechanics. Suppose a point in space is acted on by forces X, Y, Z parallel to three rectangular axes. If the point is free, its position of equilibrium will be determined by the equations $X = 0$, $Y = 0$, $Z = 0$. But if it is obliged to be on a given sur-

SECTION IV.

18. I proceed now to exhibit the application of the principles laid down in the preceding sections to some of the most important problems which occur in practice. In order to abridge the algebraical

face whose equation is $u=c$, then the condition of equilibrium is

$$Xdx+Ydy+Zdz=0$$

which, combined with the condition $du=0$, gives

$$\frac{X}{\frac{du}{dx}}=\frac{Y}{\frac{du}{dy}}=\frac{Z}{\frac{du}{dz}} \qquad (a)$$

Either of these formulæ shews that the resultant of the forces must be in the direction of a normal to the surface. The latter, considered by itself, gives the equations to the curve which would be the locus of the position of equilibrium of the point, supposing different values to be given to the parameter c. If we suppose the surface to be mechanically constructed in such a way that the parameter c may be altered by the action of a certain force; for example, if c be always equal to the distance of a point, moveable in a straight line, from a fixed point in the same; then supposing a force f to act on this moveable point in the line in which it can move, the condition of equilibrium of the whole system is

$$Xdx+Ydy+Zdz=fdc=fdu$$

where $du=\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz$, and therefore

$$\left(X-f\frac{du}{dx}\right)dx+\left(Y-f\frac{du}{dy}\right)dy+\left(Z-f\frac{du}{dz}\right)dz=0$$

in which equation the variations of x, y, z are subject to the two conditions implied by the formula (a). The reasoning might perhaps be made more clear by introducing the notation of *variations* in these latter equations, and in the corresponding equations in the text; but I prefer leaving the reader to distinguish for himself the different meanings of the differentials in the different parts of the problem.

operations, I shall have occasion to use a very simple theorem, of which it will be sufficient to give the enunciation.

Theorem. The formula

$$\frac{a}{b}=\frac{a'}{b'}=\frac{a''}{b''}=\ldots\ldots$$

involves the following as a consequence,

$$\frac{a}{b}=\frac{a'}{b'}=\ldots\ldots=\frac{a+a'+a'+\ldots.}{b+b'+b''+\ldots}$$

$$=\pm\frac{\sqrt{a^2+a'^2+a''^2+\ldots}}{\sqrt{b^2+b'^2+b''^2+\ldots}}=\frac{ma+m'a'+m''a''+\ldots}{mb+m'b'+m''b''+\ldots}$$

in which m, m', m'', ... are arbitrary.

19. *Problem.* Let $u=av+a'v'+a''v''+\ldots.$ and suppose the independent quantities v, v', v'', ... have been given respectively equal to v_0, v'_0, v''_0, ... by observations whose weights are g, g', g''... Required the *weight* of the conclusion $u=u_0$, where $u_0=av_0+a'v'_0+\ldots.$

Solution. If we attribute to u a series of values different from u_0, then v, v',.... must vary according to the law of relative equilibrium (Art. 16), i. e.

$$\frac{g(v-v_0)}{a}=\frac{g'(v'-v'_0)}{a'}=\frac{g''(v''-v''_0)}{a''}=\ldots.\quad (A)$$

Let f be the force acting on u. Then by the principle of virtual velocities (Art. 12)

$$fdu=g(v-v_0)dv+g'(v'-v'_0)dv'+\ldots..$$

in which equation v, v',.... are subject to the relations (A).

Now the formula (*A*) involves the following (Art. 18),

$$\frac{g(v-v_0)dv+g'(v'-v'_0)dv'+\ldots.}{adv+a'dv'+\ldots.}=$$

$$\frac{a(v-v_0)+a'(v'-v'_0)+\ldots.}{\frac{a^2}{g}+\frac{a'^2}{g'}+\ldots.}$$

i. e. since $a(v-v_0)+a(v'-v'_0)+\ldots=u-u_0$ and $adv+a'dv'+\ldots=du$,

$$f=\frac{u-u_0}{\frac{a^2}{g}+\frac{a'^2}{g'}+\frac{a''^2}{g''}+\ldots.}$$

Hence (Art. 8) it appears, that the force tending to restore u to the value u_0 is the same as if that value had been found by a direct observation of weight G, where

$$\frac{1}{G}=\frac{a^2}{g}+\frac{a'^2}{g'}+\frac{a''^2}{g''}+\ldots.$$

and therefore G is the weight required[f]. (*Theor. Comb.* Art. 18).

20. The most general form of the problem solved in the last article would be, to find the weight of the conclusion $u=u_0$, where u is *any* function of the quantities v, v',... and u_0 its value corresponding to $v=v_0$, $v'=v'_0$,.... In this case, let $\frac{du}{dv}=U$, $\frac{du}{dv'}=U'$, ... and let U_0, U'_0,... be the values of U, U'.... when $v=v_0$,.... Then we shall have, putting f as before for the force acting on u,

$$fdu=g(v-v_0)\,dv+g'(v'-v'_0)\,dv'+\ldots.$$

together with the equations

---

[f] The conclusion of Art. 14 is obviously a particular case of this proposition.

$$du = Udv + Udv' + \ldots .$$

and the conditions of relative equilibrium, viz.

$$\frac{g\,(v-v_0)}{U} = \frac{g'\,(v'-v'_0)}{U'} = \ldots . \qquad (A)$$

and hence, exactly as in the former case,

$$f = \frac{U(v-v_0) + U'(v'-v'_0) + \ldots .}{\dfrac{U^2}{g} + \dfrac{U'^2}{g'} + \ldots ..} \qquad (B)$$

Now in order that it may be possible to assign a determinate *weight* to the conclusion $u = u_0$, that is, to assimilate it completely to the result of a direct observation, it is necessary that the value of f should be capable of being put in the form

$$f = G.\,(u-u_0)$$

where G is some *constant* (Art. 8). It is obvious however that this will not *generally* be possible, and therefore the problem does not generally admit of solution. In other words, the conclusion $u = u_0$ is not *generally* equivalent to the result of a direct observation.

But if we suppose that the variations of v, v', ... are restricted within very narrow limits, or that the possible errors of observation are very small, and consequently neglect the powers and products of $v - v_0$, $v' - v'_0$, ... it is evident that we may, in the expression (B) write U_0 for U, &c., and since we have, to this degree of approximation, by Taylor's Theorem,

$$u - u_0 = U_0\,(v-v_0) + U'_0\,(v'-v'_0) + \ldots .$$

there results $\qquad f = G\,(u-u_0)$

where $\qquad \dfrac{1}{G} = \dfrac{U_0^2}{g} + \dfrac{U'^2_0}{g'} + \ldots ..$

and since G is now constant, it expresses the weight required (*Theor. Comb.* Art. 18).

21. It is important to observe, that if $v, v', v'', \ldots.$ instead of being independent, were connected by any given equations of condition, the force f would still be determined by the equation

$$f du = g(v-v_\circ)\,dv + g'(v'-v'_\circ)\,dv' + \ldots$$

only the variations of $v, v', \ldots.$ would no longer be expressed by the formula (A) of the last article, but by the equations obtained by eliminating as many as possible of the differentials $dv, dv' \ldots.$ from the given equations of condition combined with the equations

$$g(v-v_\circ)\,dv + g'(v'-v'_\circ)\,dv' + \ldots = o$$

and $\qquad du = o,$

and then equating separately to zero the coefficients of the remaining (independent) differentials[g].

22. If in the problems of Arts. 16, 20, and 21, we take for the function u the following, viz. (n being any positive number):

$$u = \{g(v-v_\circ)^2 + g'(v'-v'_\circ)^2 + \ldots.\}^n$$

the equation expressing the equilibrium of $v, v', \ldots.$ for any assumed value of u, namely,

$$g(v-v_\circ)\,dv + g'(v'-v'_\circ)\,dv' + \ldots = o$$

is *identically* satisfied by the supposition $u =$ con-

---

[g] The illustration in the note to Art. 17 may be easily extended to this case. Suppose the point on which the forces X, Y, Z act is obliged to move in the surface $w=o$ which has no variable parameter. If we then also subject it to be in the surface $u=c$, whose parameter c is alterable by the force f in the way

stant. So that if we are obliged to give u any value u_1, different from u_0 (in this case $u_0=o$), *any* system of values of v, v',... which satisfies the equation $u=u_1$, is in equilibrio, and there is no reason for preferring one such system to another. I shall call the different systems of values of v, v', v'',.... corresponding to any assumed value of u, systems of *equal plausibility*. If we call f the force acting on u when it has any value different from its most plausible value o, we have

$$fdu=g(v-v_0)\,dv+g'(v'-v'_0)\,dv'+\ldots\ldots(1)$$

as before, and therefore, identically,

before described, the condition of equilibrium of the whole system is as before

$$fdc=fdu=Xdx+Ydy+Zdz$$

but the variations of x, y, z are now determined by the two conditions $w=o$, and

$$X\left(\frac{du}{dy}\frac{dw}{dz}-\frac{dw}{dy}\frac{du}{dz}\right)+Y\left(\frac{du}{dz}\frac{dw}{dx}-\frac{dw}{dz}\frac{du}{dx}\right)$$

$$+Z\left(\frac{du}{dx}\frac{dw}{dy}-\frac{dw}{dx}\frac{du}{dy}\right)=o.$$

The latter equation is obtained by eliminating the differentials from the three equations

$$Xdx+Ydy+Zdz=o,\ du=o,\ dw=o.$$

It obviously expresses the condition that the resultant of the forces $X\ Y\ Z$ is in the direction of a normal to the curve of intersection of the two surfaces $u=c$, $w=o$; and, combined with $w=o$, it determines the curve, which is the locus of the position of equilibrium of the point acted on by these forces, when the parameter c is made to vary.

The only limit to mechanical illustrations of this kind consists in the restriction imposed upon the number of variables by the dimensions of space. The problems in the text are of course subject to no such restriction.

$$f=\frac{1}{2nu^{\frac{n-1}{n}}} \qquad (2)$$

The condition of ***relative equilibrium*** being here satisfied identically, the variations of v, v', ... in equation (1) are absolutely independent, unless these quantities are subject to given equations of condition. (See the next Article.)

If we put $n=\frac{1}{2}$, or

$$u=\sqrt{g(v-v_0)^2+g'(v'-v'_0)^2+\ldots.} \qquad (3)$$

then $f=u$. Or in this case the ***weight*** of the conclusion $u=o$ is 1[h]. And no other value of n gives a ***weight*** for this conclusion at all.

If we put $n=1$, or

$$u=g(v-v_0)^2+g'(v'-v'_0)^2+\ldots. \qquad (4)$$

---

[h] This result appears at first sight absurd; but the difficulty disappears if we make the equation (3) *homogeneous* by assuming an arbitrary constant γ of the same kind as g, g',... and supposing

$$u=\sqrt{\frac{g}{\gamma}(v-v_0)^2+\frac{g'}{\gamma}(v'-v'_0)^2+\ldots}$$

Substituting for du its value in equation (1), we get $f=\gamma.\,u$; or the weight of the conclusion $u=o$, is γ. In like manner the general equation might be made homogeneous, by introducing an arbitrary quantity a, of the same kind as v, and assuming

$$u=a\left\{\frac{g}{\gamma}\,\frac{(v-v_0)^2}{a^2}+\frac{g'}{\gamma}\,\frac{(v'-v'_0)^2}{a^2}+\ldots\right\}^n$$

in which case we should have, instead of equation (3),

$$f=\frac{\gamma a}{2n}\left(\frac{a}{u}\right)^{\frac{n-1}{n}}$$

This being well understood, it saves trouble to use the non-homogeneous equations.

then $f=\frac{1}{2}$, or there is a *constant force* tending to diminish u when any value different from o is given to it.

23. The conclusions of the last article are independent of any supposition as to the existence or nonexistence of relations between the quantities v, v', v'', ...

If these quantities are independent, then the most plausible value of v is $v_\circ$, of v', $v'_\circ$, ... But if they are subject to given equations of condition, which make it impossible for them to receive these values simultaneously, then, since we have seen that if

$$u=g\,(v-v_\circ)^2+g'\,(v'-v'_\circ)^2+\ldots..$$

there is a *constant force* tending to diminish u as long as it is different from o, it is plain that $v, v', \ldots$ must be so assumed as to give to this function (which is essentially positive) its *least possible* value. This is the principle of *least squares;* of which we shall see a much more direct proof immediately.

SECTION V.

24. *Problem.* Let V, V', V'', ... be given functions of the independent variables $x, y, z, \ldots.$ (not more in number than the functions themselves). And suppose that direct observations whose weights are $g, g', g'', \ldots.$ have given the values K, K', K'' ..... for V, V', V''...... It is required to assign the most plausible values of x, y, z, ... and the weight of the corresponding determination of any given function of those quantities.

The principles of Arts. 8 and 12 give at once for the equation of equilibrium,

$$g(V-K)\,dV+g'(V'-K')\,dV'+\ldots\quad =o$$

which may be otherwise written

$$\frac{1}{2}\,d\Omega=o$$

if $\quad \Omega=g(V-K)^2+g'(V'-K')^2+\ldots.$

This is the well known equation which shews that the most plausible values of x, y, z,... are those which make the function Ω a minimum; whence this solution has received the name of the method of least squares.

Since x, y, z, ... are independent, the final equations for determining them will be

$$\frac{d\Omega}{dx}=0,\ \frac{d\Omega}{dy}=0,\ \frac{d\Omega}{dz}=0.....$$

25. If the number of functions V, V', . . . be greater than that of the independent variables x, y, z, ... then V, V', ... are subject to *equations of condition*, and, the observations being not absolutely accurate, the minimum value of Ω, which we will denote by Ω_o, will generally be different from 0.

We have seen (Arts. 22, 23) that there is a constant force, represented by $\frac{1}{2}$, tending to diminish Ω, which is counteracted by the relations between V, V', ..., when Ω has its minimum value. Let X, Y, Z, . . . be the forces acting on x, y, z, ... then by the principle of virtual velocities,

$$\tfrac{1}{2}d\Omega=Xdx+Ydy+Zdz+\ldots$$

in which equation dx, dy, dz, ... are absolutely independent, since the condition of *relative equilibrium* (Arts. 16, 21, 22) subsists identically; and consequently,

$$X=\tfrac{1}{2}\frac{d\Omega}{dx}, \; Y=\tfrac{1}{2}\frac{d\Omega}{dy}, \; Z=\tfrac{1}{2}\frac{d\Omega}{dz}, \ldots.$$

Hence it appears that the equations at the end of the last article signify that in the state of equilibrium of the system the force acting on each of the *independent* variables is separately equal to nothing; as of course we might have assumed *à priori.*

26. Let x_0, y_0, z_0, ... be the most plausible values of x, y, z, ... or the values which satisfy the equations $X=0$, $Y=0$, $Z=0$, ... If we disturb the equilibrium of the system by giving to x a value different from x_0, the values of the other variables will in general be changed also; the law of their variation being that of *relative equilibrium*[i] (Art. 16.) which expresses that for each assumed value of x, the values of y, z, ... are in equilibrio among themselves, and which, since these variables are independent, consists in the equations

$$Y=0, \; Z=0, \; \ldots..$$

If therefore we call p the force acting on x, in consequence of a disturbance of this kind, we have $p=X$, together with the equations just written, by means of which y, z, ... may be eliminated, and the value of p found in terms of x only. If the result should be of the form

---

[i] Such a disturbance would be actually produced, for example, by a direct observation of x, giving a value different from x_0; or by the discovery of the *true value* of x; supposing in either case that no new information is obtained concerning y, z ....

$$p = C \cdot (x - x_\circ),$$

then the constant C would be (Art. 8) the *weight* of the determination $x = x_\circ$. But if not, then on the supposition that the possible errors of observation are *small*, the weight in question will be represented by the *limit* of the expression $\dfrac{X}{x - x_\circ}$, or, which is the same thing, by the value of $\dfrac{dX}{dx}$ corresponding to $x = x_\circ$[k]. The weights of the determinations of the values of $y, z, \ldots.$ will be found in the same way.

27. Let u be any given function of $x, y, z, \ldots.$, and $u_\circ$ its value corresponding to $x_\circ, y_\circ, z_\circ, \ldots.$, then $u_\circ$ is the most plausible value of u. If we now suppose the system to be disturbed by giving

---

[k] The following may be taken as an illustration of this subject; and the reader will have no difficulty in supplying a similar one for the more general problem of the next article. Suppose a point acted on by forces X, Y, Z, which are functions of its coordinates. Its position of equilibrium will be determined by the equations $X=0$, $Y=0$, $Z=0$. Suppose now it is forced to be in a plane parallel to the plane of yz; its new position of equilibrium will be determined by the equations $x=x'$, $Y=0$, $Z=0$, where x' is the abscissa of the plane. And the force of restitution, or the pressure on the plane, will be given by the corresponding value of X. Also the equations $Y=0$, $Z=0$, by themselves, express the law of *relative equilibrium*; or, they are the equations to the locus of the position of equilibrium of the point, when the plane is supposed to move, preserving its parallelism to the plane of xy.

to u a value different from u_0, and call f the force acting on u, we have, since the force acting on Ω is $\frac{1}{2}$,

$$fdu = \tfrac{1}{2}d\Omega\,; \qquad (1)$$

in which equation the variables are subject to the conditions of relative equilibrium, namely, (since X, Y, Z, ... are the forces acting on x, y, z, ...)

$$\frac{X}{\frac{du}{dx}} = \frac{Y}{\frac{du}{dy}} = \frac{Z}{\frac{du}{dz}} = \ldots . \qquad (A)$$

These conditions being supposed, the value of f is $\frac{1}{2}\,\frac{d\Omega}{du}$, or, which is the same thing,

$$f = (u-u_0)\,\frac{d\Omega}{d\,.\,(u-u_0)^2} \qquad (2)$$

an equation which shews (Article 8) that if $\dfrac{d\Omega}{d\,.\,(u-u_0)^2}$ is constant, (which will not generally be the case,) then its value is the *weight* of the determination $u = u_0$. In all cases we may take the *limit* of this expression, or, which is the same thing, the limit to which $\dfrac{\Omega-\Omega_0}{(u-u_0)^2}$ converges when u converges to u_0 (subject to the conditions (A)), to represent the weight in question so long as all the variations are very small.

28. If we adopt the subscript o to indicate the value of any function corresponding to x_0, y_0, z_0, ... and if moreover we suppose x, y, z, ... to be expressed as functions of X, Y, Z, ... then, observing that $X_0 = 0$, $Y_0 = 0$ .... we have, as far as terms of the first order, by Taylor's theorem,

$$x - x_0 = \left(\frac{dx}{dX}\right)_0 X + \left(\frac{dx}{dY}\right)_0 Y + \dots .$$

and therefore, as far as terms of the second order,

$$(u-u_0)^2 = \left\{ \left(\frac{du}{dx}\right)_0 (x-x_0) + \left(\frac{du}{dy}\right)_0 (y-y_0) + \dots \right\}^2$$

$$= \left\{ \left(\frac{du}{dx}\right)_0 \left(\frac{dx}{dX}\right)_0 X + \left(\frac{du}{dy}\right)_0 \left(\frac{dy}{dY}\right)_0 Y + \dots .\right.$$

$$\left. + \left(\frac{du}{dx}\right)_0 \left(\frac{dx}{dY}\right)_0 Y + \left(\frac{du}{dy}\right)_0 \left(\frac{dy}{dX}\right)_0 X + \dots . \right\}^2$$

and, as far as terms of the same order,

$$\Omega - \Omega_0 = X(x-x_0) + Y(y-y_0) + \dots .[1]$$

$$= X^2 \left(\frac{dx}{dX}\right)_0 + Y^2 \left(\frac{dy}{dY}\right)_0 + \dots$$

$$+ XY \left(\frac{dx}{dY} + \frac{dy}{dX}\right)_0 + \dots .$$

If we now divide the value of $\Omega - \Omega_0$ by that of $(u-u_0)^2$, it is obvious that the limiting value of the quotient, when $x, y, \dots$ converge to $x_0, y_0, \dots .$, subject to the conditions (A), will be found by merely writing $\left(\frac{du}{dx}\right)_0, \left(\frac{du}{dy}\right)_0, \dots .$ for $X, Y, \dots .$ in every term, since both numerator and denominator are of the same degree with respect to $X, Y, \ldots.$ But after this substitution the value of $(u-u_0)^2$ becomes identical with that of $(\Omega-\Omega_0)^2$. And therefore calling G the limit in question, we have

---

[1] For, since $X_0 = 0$ &c., $X = \left(\frac{dX}{dx}\right)_0 (x-x_0)$ &c. and $\left(\frac{dX}{dx}\right)_0 = \frac{X}{x-x_0}$; whence $\Omega - \Omega_0 = \frac{1}{2}\left(\frac{d^2\Omega}{dx_2}\right)_0 (x-x_0)^2 + \dots = \left(\frac{dX}{dx}\right)_0 (x-x_0)_2 + \dots . = X(x-x_0) + \dots .$ This equation, which subsists generally only for small variations, is true without restriction when the functions $V, V' \dots .$ are all linear, as we shall see afterwards. (Art. 46. Equation (2)).

$$\frac{1}{G}=\left(\frac{du}{dx}\right)_0^2\left(\frac{dx}{dX}\right)_0+\left(\frac{du}{dy}\right)_0^2\left(\frac{dy}{dY}\right)_0+\ldots.$$
$$+\left(\frac{du}{dx}\right)_0\left(\frac{du}{dy}\right)_0\left(\frac{dx}{dY}+\frac{dy}{dX}\right)_0+\ldots.$$

Or, the reciprocal of the weight of the determination $u=u_0$ is found by expressing $\Omega-\Omega_0$ in terms of $X, Y, Z, \ldots.$ and then writing $\left(\frac{du}{dx}\right)_0, \left(\frac{du}{dy}\right)_0, \ldots.$ instead of these quantities in the value so expressed.

I have introduced this investigation chiefly in order to shew the coincidence of the result with that obtained by Gauss (*Theor. Comb.* 29.) in the particular case of linear functions, which will be considered further on. It is proper to remark, however, that there is no real distinction between the cases, since it is assumed that the *small* variations of any function follow the law of linearity.

29. In the case in which the expression $\frac{d\Omega}{d(u-u_0)^2}$ obtains a constant value G by virtue of the conditions (A) Art. 27, we have without restriction $f=G.(u-u_0)$ for the force acting on u when the system is disturbed by giving that variable a value different from u_0, and G is the weight of the determination $u=u_0$. Also we have at once by integration

$$\Omega-\Omega_0=G\,.\,(u-u_0)^2 \qquad (1)$$

an equation which is not *identical,* but subsists only in conjunction with the conditions (A). (The same equation is obviously true with the same restriction in *all* cases for *small variations,* G having the same meaning as in the last article.) It is easily seen that it gives the *relative minimum* of Ω, i. e. its least value consistent with any particular value of u; for the conditions (A) express the *coexistence* of the equations $d\Omega=0$, $du=0$.

If we suppose Ω_0+E to be the greatest value which can possibly be attributed to Ω, then the true value of u must be between the limits $u_0 \pm \sqrt{\frac{E}{G}}$. (*Theor. Comb.* Art. 30.)

30. Since the equation of equilibrium

$$Xdx+Ydy+Zdz+\ldots=0,$$

is identically satisfied by the supposition Ω = constant, it follows that if we were obliged to give Ω any particular value Ω_0+E different from Ω_0, there would be no reason for preferring any one system of values of the variables consistent with this value of Ω, to any other. All systems which give the same value to Ω are therefore systems of *equal plausibility.* Moreover on the same supposition (that Ω_0+E is the *true* value of Ω), the value of u (see the last article) would be in general completely indeterminate between the limits $u_0 \pm \sqrt{\frac{G}{E}}$, since it would be generally possible to give u any value between these limits without contradicting the condition $\Omega=\Omega_0+E$.

31. If in Art. 27 we put $u=\sqrt{\Omega-\Omega_0}$, then

$u_\circ = 0$, the conditions (A) are satisfied identically, and we have at once

$$\frac{d\Omega}{d\,.\,(u-u_\circ)^2} = 1.$$

or the weight of the conclusion $\sqrt{\Omega - \Omega_\circ} = 0$ is unity. If therefore we know *à priori*, or discover *à posteriori*, by means of the observations themselves, (the mode of doing which will be explained afterwards,) that there is reason to apprehend an error of $\pm\epsilon$ in a determination whose weight is unity, it will follow that the value to be *feared* for Ω is $\Omega = \Omega_\circ + \epsilon^2$. This equation may be considered as the expression of the *uncertainty* or *want of precision* of the result of the observations. Also if u be any function whose most plausible value has been found $= u_\circ$ with weight G, the error to be feared in this determination will be $\pm \frac{\epsilon}{\sqrt{G}}$, (see Art. 15,) which agrees with the conclusions of the last Article.

32. The conclusions of this section admit of a simple geometrical interpretation when there are only three independent variables. In this case let x, y, z, be rectangular coordinates of a point in space whose position is to be determined by the observations. The most plausible position is the point ($x_\circ$, $y_\circ$, $z_\circ$,) which we will call O, (determined by the equations $X = 0$, $Y = 0$, $Z = 0$,) in which the forces arising from the observations balance one another. If it be placed in any position different from O, it is acted on by a force whose components are X, Y, Z, parallel to the three axes. The equation

$\Omega = \Omega_0 + C$ will represent, if we give different positive values to C, a system of similar surfaces disposed round the point O, (which point is the locus of this equation in the case in which $C=0$). Each of these surfaces is a surface of *equal plausibility*. In fact, since X, Y, Z are proportional (Art. 25.) to the partial diff. coefficients of Ω, it follows that if the point be placed anywhere on one of these surfaces, the force of restitution acting upon it is in the direction of the normal; so that if it were forced to be in that particular surface, it would be in equilibrio indifferently at any point in it, and there would be no reason for placing it at one point rather than another.

If we suppose the value of a function u to be given $= c$, in other words, if the point be obliged to be in a given surface whose equation is $u = c$, it cannot be in equilibrio unless the force arising from the observations be in the direction of a normal to this latter surface. Its position of equilibrium will therefore be determined by the equations

$$\frac{X}{\frac{du}{dx}} = \frac{Y}{\frac{du}{dy}} = \frac{Z}{\frac{du}{dz}} \qquad \text{(A)}$$

combined with $u=c$; and will obviously be at that point of the surface $u=c$ in which it is touched by that one of the system of surfaces $\Omega = \Omega_0 + C$, which *only just* touches it. The equations (A) by themselves determine the locus of this point when the parameter c is supposed to vary.

If we denote by ϵ the same thing as in Art. 31, then the general uncertainty (or want of precision) of the result may be described by saying that the observations determine, instead of a *point*, a *space* bounded by the surface whose equation is $\Omega = \Omega_0 + \epsilon^2$.

SECTION VI.

33. The principles already established enable us to find (in the general problem of the last section) the most plausible value of any proposed function of the unknown quantities, as well as the weight of such determination. But this weight merely expresses the number of concurrent observations of the kind assumed as a standard, to which the determination is equivalent. And therefore in order that we may know what confidence it is actually entitled to, it is necessary that we should know what error is to be apprehended in a determination of given weight. If this is not known *à priori*, it must be estimated by means of the observations themselves; and it is obvious that this can only be done by observing in some way the degree in which they contradict given conditions.

34. If the number of observations (or of the functions $V, V', \ldots.$) be not greater than that of the unknown quantities $x, y, z, \ldots.$, then there can generally be no inconsistency in the results, and the estimation *à posteriori* of the errors fails entirely. But suppose σ to be the number of observations and τ the number of unknown quantities; then if σ be greater than τ, we can generally obtain $\sigma - \tau$ essentially different equations of condition between the σ quantities $V, V', V'', \ldots.$ by eliminating $x, y, z, \ldots.$. And this may generally be done in an infinite variety of ways.

Now each equation of condition, being of the form

$$u = F(V, V', V'', \ldots) = o$$

gives the *true* value o, of a certain function u of the unknown quantities. On the other hand, supposing $K, K', K'', \ldots$ the values of $V, V', V'', \ldots$ given by the observations, the value of u deduced directly from them, (without taking into account the equations of condition) is $F(K, K', K'', \ldots)$, and the *weight* of this deduction can be found (Arts. 19, 20). Thus we have at once the *actual errors* of $\sigma - \tau$ determinations of *known weights*, and consequently $\sigma - \tau$ examples of the actual value of the product *weight* × $(\textit{error})^2$ The *average value* of this product deduced from the known errors of an infinite number of actual and *independent* observations would give the value to be reasonably feared in any particular case[m]. Now as the observations only furnish $\sigma - \tau$ examples, we must be content with the average deduced from them; and, of course, the greater the number $\sigma - \tau$ is, the greater will be the right of this average to represent that of an *infinite* number. But since the $\sigma - \tau$ functions u may generally be chosen in an infinite variety of ways, it is necessary to inquire *how* they must be chosen in order that their values deduced as above may be really equivalent to the results of *independent* observations.

35. The essential property of independent observations is that the result of any one does not alter or modify in any way the result of any other, so long as no account is taken of any existing relations between the quantities observed. In this sense all

---

[m] The square of the "error medius metuendus" of Gauss.

actual observations are independent. In the case before us, for instance, V, V', V'',... have been given by actual observations equal to K, K'... respectively; and, if we take no account of the equations of condition, K would continue to be the most plausible value of V, notwithstanding any alterations which might be made in the values of any of the other quantities in consequence (suppose) of new observations of them, or the discovery of their *true* values.

But if u and v be any two functions of V, V',...., and u_1, v_1, their values corresponding to $V=K$, $V'=K'$,... (i.e. their most plausible values as given immediately by the observations), then if any new value were given to u, in consequence of a direct observation of that quantity, or of the discovery of its true value, v_1 would generally cease to be the most plausible value of v. So that the determinations $u=u_1$, $v=v_1$ are not generally entitled to be considered as the results of *independent* observations. The question then is, what relation must subsist between the functions u and v, in order that the most plausible value of one of them may not be altered by any direct change in the value of the other.

36. The only case of this problem which it is important to consider, is that in which the functions u and v are both *linear*. Suppose

$$u=lV+l'V'+l''V''+\dots.$$
$$v=mV+m'V'+m''V''+\dots.$$

then we have

$$u_1=lK+l'K'+\dots. \qquad (\alpha)$$
$$v_1=mK+m'K'+\dots. \qquad (\beta)$$

and if $g, g', \ldots$ be the weights of the original observations, the weights (G and H) of the two determinations (α) and (β) are given by the equations (Art. 19)

$$\frac{1}{G} = \frac{l^2}{g} + \frac{l'^2}{g'} + \ldots$$

$$\frac{1}{H} = \frac{m^2}{g} + \frac{m'^2}{g'} + \ldots$$

Now if we assumed for u a value different from u_1, the variations of V, V', V'', ... would be subject to the law of relative equilibrium (Art. 16), namely

$$\frac{g(V-K)}{l} = \frac{g'(V'-K')}{l'} = \frac{g''(V''-K'')}{l''} = \ldots.$$

which implies (Art. 18)

$$\frac{l(V-K) + l'(V'-K') + \ldots}{\frac{l^2}{g} + \frac{l'^2}{g'} + \ldots} = \frac{m(V-K) + m'(V'-K') + \ldots}{\frac{lm}{g} + \frac{l'm'}{g'} + \ldots}$$

that is

$$v - v_1 = G \,.\, \left(\frac{lm}{g} + \frac{l'm'}{g'} + \ldots\right) \,.\, (u - u_1)$$

so that if u be different from u_1, v will be also different from v_1, *unless*

$$\frac{lm}{g} + \frac{l'm'}{g'} + \frac{l''m''}{g''} + \ldots \quad = 0 \qquad (I)$$

which is therefore the condition of *independence*[n].

---

[n] It is easily seen that the same condition would follow immediately from the principles of Art. 18 of the *Theoria Combinationis.*

If the weights g, g', ... were all equal, this would reduce itself to

$$lm + l'm' + l''m'' + \ldots \quad = 0.$$

37. If the functions u, v, were not linear, these conclusions would still subsist for *small* variations. We should only have to write instead of l, m, ... the values of the partial differential coefficients $\frac{du}{dV}, \frac{dv}{dV}, \ldots$ corresponding to $V = K$, $V' = K'$....

38. To return to the subject of Art. 34, it appears that if we can find $\sigma - \tau$ linear equations of condition between the quantities V, V', V'', ... such that, taken two and two, their coefficients satisfy the conditions of independence (I), Art. 36, we shall then have the actual errors of $\sigma - \tau$ determinations equivalent to as many *independent observations* of known weight. And the average value of the product *weight* × (*error*)^2 deduced from these examples, may be taken as the value which is to be feared in any determination of which the error is not given, with so much the more confidence as the number $\sigma - \tau$ is more considerable. I shall shew hereafter, that when the functions V, V', ... are all linear with respect to x, y, z, ... an infinite number of such systems of $\sigma - \tau$ equations can in general be found; but that each system gives *the same average value* of the product in question, and that this value is $\frac{\Omega_0}{\sigma - \tau}$, where Ω_0 has the same meaning as in the last section.

In the case in which $\sigma - \tau = 1$, there is only *one* system (consisting also of only one equation); and if $\sigma = \tau$ there are *no* equations of condition, and the whole process fails; in fact $\frac{\Omega_0}{\sigma-\tau}$ then takes the indeterminate form $\frac{0}{0}$.

It will follow (Art. 31) that the *want of precision* of the general result is expressed by the equation

$$\Omega = \left(1 + \frac{1}{\sigma - \tau}\right)\Omega_0.$$

SECTION VII.

39. The only case of the problem considered in the fifth section, which usually occurs in practice, is that in which the functions V, V', ... are all linear. For if, as is generally the case, the quantities to be determined are so nearly known that the powers and products of the errors of their approximate values may be neglected, we may take as the unknown quantities of the problem the *corrections* to be applied to these approximate values; and the differences between the observed values of the functions and their approximate calculated values will be linear functions of the corrections required, the coefficient of each correction being that value of the partial diff. coeff. of the function which corresponds to the approximate values of all the unknown quantities.

40. Suppose then

$$\sqrt{g}.(V-K) = v,\ \sqrt{g'}.(V'-K') = v' \text{ \&c.}$$

Since an observation of V whose weight is g is equivalent (Art. 14) to an observation of $\sqrt{g}\,.\,V$ whose weight is unity, we may consider the equations $v=0$, $v'=0$, $v''=0$, ... to have been given by observations with weights respectively $=1$. Also, since the functions V, V', ... are now supposed linear, we may assume

$$\begin{aligned} v &= ax + by + cz + \ldots + k \\ v' &= a'x + b'y + c'z + \ldots + k' \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

Moreover we shall now have

$$\Omega = v^2 + v'^2 + v''^2 + \ldots.$$

and, putting $a^2 + a'^2 + \ldots = \Sigma(a^2)$ &c.,

$$\left.\begin{aligned} \tfrac{1}{2}\frac{d\Omega}{dx} &= X = \Sigma(a^2)x + \Sigma(ab)y + \Sigma(ac)z + \ldots + \Sigma(ak) \\ \tfrac{1}{2}\frac{d\Omega}{dy} &= Y = \Sigma(ab)x + \Sigma(b^2)y + \Sigma(bc)z + \ldots + \Sigma(bk) \\ \tfrac{1}{2}\frac{d\Omega}{dz} &= Z = \Sigma(ac)x + \Sigma(bc)y + \Sigma(c^2)z + \ldots + \Sigma(ck) \end{aligned}\right\}(1)$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

The most plausible values x_0, y_0, z_0, . . . of the unknown quantities are those which satisfy the equations $X = 0$, $Y = 0$, $Z = 0$, ...

41. Suppose that by the ordinary process of elimination the values of x, y, z, ... were found from equations (1) as linear functions of X, Y, Z, ... it is evident that the *constant* term in each would be its most *plausible value*, and we should have

$$\begin{aligned} x &= x_0 + [\alpha\alpha]\,.\,X + [\alpha\beta]\,.\,Y + [\alpha\gamma]\,.\,Z + \ldots \\ y &= y_0 + [\beta\alpha]\,.\,X + [\beta\beta]\,.\,Y + [\beta\gamma]\,.\,Z + \ldots. \\ z &= z_0 + [\gamma\alpha]\,.\,X + [\gamma\beta]\,.\,Y + [\gamma\gamma]\,.\,Z + \ldots. \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

in which $[\alpha\alpha]$, $[\alpha\beta]$, ... merely signify constant coefficients, the first letter indicating the rank of the equation to which the coefficient belongs, and the second, the rank of the coefficient in the equation.

42. The *weights* of the determinations $x=x_0$, $y=y_0$, $z=z_0$, ... are the reciprocals of the coefficients $[\alpha\alpha]$, $[\beta\beta]$, $[\gamma\gamma]$,.... respectively. For X, Y, Z, ... express (Art. 25) the forces acting on x, y, z, ... If therefore we give to x any value different from x_0, the conditions of relative equilibrium of the other independent variables are (as in Art. 26) $Y=0$, $Z=0$, ... and the first of the equations of the last article gives, supposing these conditions,

$$X=\frac{x-x_0}{[\alpha\alpha]}$$

for the force acting on x; whence it follows (Art. 8) that $\frac{1}{[\alpha\alpha]}$ is the weight of the determination $x=x_0$. In like manner, if the equilibrium of the system were disturbed by an alteration of the value of y, the relative equilibrium of x, z,.... would be expressed by the equations $X=0$, $Z=0$,... and the second equation of Art. 41 would give $Y=\frac{y-y_0}{[\beta\beta]}$, whence, $\frac{1}{[\beta\beta]}$ is the weight of the conclusion $y=y_0$, and so for the rest. (*Theor. Comb.* Art. 21.)

43. It is necessary for the consistency of the theory, to inquire whether the solution of the equations $X=0$, $Y=0$,...... will always give determinate values for the unknown quantities; or, which is the same thing, whether the elimination supposed in Art. 41 is always possible. I shall therefore consider this point before proceeding further. Now the elimination in question is always possible if the functions X, Y, Z,... are mutually independent, i. e. if there exist no identical equa-

tions of condition between them. If there be any such identical relations, it is easily seen that there must exist one or more *linear* equations. Suppose then that there is an identical linear equation, such as

$$lX + mY + nZ + \dots\dots = \frac{1}{2}h \qquad (\alpha)$$

Since $X = \frac{1}{2}\frac{d\Omega}{dx}$, &c., it follows that Ω must satisfy the partial differential equation

$$l\frac{d\Omega}{dx} + m\frac{d\Omega}{dy} + \dots = h \qquad (\beta)$$

of which the general solution is

$$\Omega = h\frac{lx + my + nz + \dots}{l^2 + m^2 + n^2 + \dots} + F(mx - ly, ny - mz, \dots\dots)^{\circ}$$

Now if the observations were absolutely accurate, it would necessarily be possible to satisfy the equations $X = 0$, $Y = 0, \dots$ all at once, since they are merely particular combinations of the equations $v = 0$, $v' = 0, \dots\dots$ (namely $X = \Sigma(av)$, $Y = \Sigma(bv) \dots$), which could not then be inconsistent with each other. We see therefore from equation (α), that on this supposition h must $= 0$, and Ω must be of the form

$$\Omega = F(mx - ly,\ ny - mz, \dots\dots\ ) \qquad (\Omega)$$

whence it follows immediately that Ω would remain

° The number of the terms $mx - ly$, $ny - mz$, ... is equal to that of the variables; the differences being taken by writing x, y, z, ... in any order, and combining the 1st and 2nd, 2nd and 3rd, 3rd and 4th, ... last and first.

constant if, $\alpha, \beta, \gamma, \ldots.$ being any arbitrary constants, $x, y, z, \ldots$ varied subject to the conditions

$$\frac{x+\alpha}{l} = \frac{y+\beta}{m} = \frac{z+\gamma}{n} = \ldots.$$

Consequently the equation $\Omega = 0$, which is equivalent to the system of equations $v = 0$, $v' = 0$, $v'' = 0, \ldots.$ would be *insufficient* to determine x, $y, z, \ldots.$ It appears then that the required elimination cannot fail, unless the functions $v, v', \ldots.$ are so constituted that the observations would be insufficient to determine the unknown quantities even supposing them accurate; a case which, of course, never occurs in practice.

It may be observed, that the equation (β) in fact leads to the same conclusion directly, since the suppositions $h = 0$ and $\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n} = \ldots$ are sufficient to reduce that equation to $d\Omega = 0$, or $\Omega =$ constant. (*Theor. Comb.* Art. 23.)

44. Since the function Ω, from its nature, necessarily admits of *a minimum value*, it must in fact be *always* possible to satisfy the equations $X = 0$, $Y = 0, \ldots.$ all at once, whether they are sufficient to determine a *single system* of values of $x, y, z, \ldots$ or not. And therefore the constant h of the last article must equal 0, whatever be the errors of the observations, and not merely on the hypothesis of their accuracy. This shews that if *accurate* observations would be insufficient, they can never *appear* to become sufficient in consequence of errors.

It is worth observing, that since the equation (Ω) (Art. 43) represents in all cases the form of Ω on the supposition that there exists an identical linear equation $lX+mY+,..=0$, if we suppose only three unknown quantities, and consider them as coordinates of a point in space, the equation $\Omega=\Omega_0+\epsilon^2$, which, if ϵ be arbitrary, represents (Art. 32) any surface of *equal plausibility*, becomes

$$F(mx-ly,\ ny+mz,\ lz-nx)=\Omega_0+\epsilon^2$$

which is the general equation to *cylindrical* surfaces. In fact the equation $lX+mY+nZ=0$ merely expresses the condition that the normal is always at right angles to a fixed line. Let x_0, y_0, z_0, be any system of values which satisfy the equation $\Omega=\Omega_0$, then since the value of Ω is unaltered as long as x, y, z, are subject to the condition

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$$

it follows that this formula represents the most plausible position of the point (x, y, z); which is therefore not a determinate *point*, but a determinate *straight line*, and is obviously the axis of all the cylindrical surfaces of equal plausibility.

In like manner, if we supposed *two* equations of condition, the normal to the surface $\Omega=\Omega_0+\epsilon^2$ must be at right angles to *two* given lines, i. e. to *one* given plane. And therefore the surface in question (since Ω is of the 2nd degree) must consist of two *parallel planes*. In fact, since the two linear equations are equivalent to an expression of the form $\frac{X}{\lambda}=\frac{Y}{\mu}=\frac{Z}{\nu}$, it is very easily seen[p] that the

---

[p] For we have

$$\frac{X}{\lambda}=\frac{Y}{\mu}=\frac{Z}{\nu}=\frac{Xdx+Ydy+Zdz}{\lambda dx+\mu dy+\nu dz}=\frac{\frac{1}{2}d\Omega}{d(\lambda x+\mu y+\nu z+\kappa)}=\frac{d\Omega}{dw}$$

suppose. Therefore $d\Omega=\frac{X}{l}dw$, whence X must be a function form of Ω can be no other than

$$\Omega_0 + A^2\left(\lambda(x-x_0) + \mu\,(y-y_0) + \nu\,(z-z_0)\,\right)^2,$$

so that the equation $\Omega = \Omega_0 + \epsilon^2$ resolves itself into

$$A\left(\lambda\,(x-x_0) + \mu\,(y-y_0) + \nu\,(z-z_0)\,\right) = \pm\epsilon$$

which represents two parallel planes at equal distances from a *plane of greatest plausibility* corresponding to $\epsilon = 0$ (x_0, y_0, z_0, signify as before, any system of values which satisfy the equation $\Omega = \Omega_0$).

45. We will now return to the general subject.

Since $$\frac{1}{2}\frac{d\Omega}{dX} = \frac{1}{2}\frac{d\Omega}{dx}\,\frac{dx}{dX} + \frac{1}{2}\frac{d\Omega}{dy}\,\frac{dy}{dX} + \ldots.$$

$$= [\alpha\alpha]\,X + [\beta\alpha]\,Y + [\gamma\alpha]\,Z + \ldots \quad \text{(Art. 41.)}$$

and in like manner

$$\frac{1}{2}\frac{d\Omega}{dY} = [\alpha\beta]\,X + [\beta\beta]\,Y + [\gamma\beta]\,Z + \ldots$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

the conditions $\dfrac{d^2\Omega}{dXdY} = \dfrac{d^2\Omega}{dYdX}$, &c. give

$$[\alpha\beta] = [\beta\alpha],\ [\beta\gamma] = [\gamma\beta],\ \text{\&c.}$$

(Other properties of these coefficients may be seen in the *Theor. Comb.*)

46. Since $\Sigma\,(a^2)\,x_0 + \Sigma\,(ab)\,y_0 + \ldots. + \Sigma\,(ak) = 0$, the value of X may be written (Art. 40)

$$X = \Sigma\,(a^2)\;(x-x_0) + \Sigma\,(ab)\;(y-y_0) + \ldots$$

of w, and must also be of the first degree, since Ω is of the 2nd. The same must be true of Y and Z. So that we may obviously assume, (κ being arbitrary,) $\frac{X}{\lambda} = \frac{Y}{\mu} = \frac{Z}{\nu} = C.w$ whence $d\Omega = C\,.\,wdw$, and the proposition in the text follows immediately.

or, putting $x - x_0 = x', \; y - y_0 = y', \dots.$

$$X = \Sigma\,(a^2)\,x' + \Sigma\,(ab)\,y' + \Sigma\,(ac)\,z' + \dots.$$

and similarly

$$Y = \Sigma\,(ab)\,x' + \Sigma\,(b^2)\,y' + \Sigma\,(bc)\,z' + \dots$$

$$Z = \Sigma\,(ac)\,x' + \Sigma\,(bc)\,y' + \Sigma\,(c^2)\,z' + \dots$$

&c. and these are the values of the partial diff. coefficients of $\frac{1}{2}\Omega$ taken with respect to x, y, z,... or, which is the same thing, with respect to x', y', z',... Hence, integrating, and observing that $\Omega = \Omega_0$ corresponds to $x' = 0$, $y' = 0$,... we get

$$\Omega = \Omega_0 + \Sigma(a^2)x'^2 + \Sigma(b^2)y'^2 + \dots + 2\Sigma(ab)x'y' + \dots(1)$$

an equation which it is otherwise easy to prove directly. It is evidently equivalent to the following, viz.

$$\Omega = \Omega_0 + Xx' + Yy' + Zz' + \dots. \quad (2)$$

Also, putting v_0, v'_0,....[q] for the values of v, v'... corresponding to $x = x_0$,.... since $X = \Sigma(av)$, $Y = \Sigma\,(bv)$, ... we have $\Sigma(av_0) = 0$, $\Sigma(bv_0) = 0$,.... and therefore, multiplying the equations

$$v = ax + by + cz + \dots \; + k$$

$$v' = a'x + b'y + c'z + \dots \; + k'$$

&c. by v_0, v'_0, v''_0, ... respectively, and adding,

$$vv_0 + v'v'_0 + v''v''_0 + \dots = \Sigma\,(v_0k)$$

consequently (since $\Sigma\,(v_0{}^2) = \Omega_0$),

$$\Omega_0 = \Sigma\,(v_0k)$$

and

$$\Sigma\,(vv_0) = \Omega_0 \qquad (3)$$

---

[q] These values must be carefully distinguished from those represented by the same notation in section IV.

If we multiply this equation by 2, and subtract it from the following, viz.

$$\Sigma(v^2)+\Sigma(v_0)^2=\Omega+\Omega_0,$$

we find

$$\Omega=\Omega_0+(v-v_0)^2+(v'-v'_0)^2+(v''-v''_0)^2+\dots \qquad (4)$$

which exhibits the condition of a *minimum* in the most obvious form. (*Theor. Comb.* Art. 26.)

Another expression for Ω may be obtained by integrating the values of $\frac{d\Omega}{dX}$, $\frac{d\Omega}{dY}$, ... found in Art. 45, viz. (observing that $\Omega=\Omega_0$ corresponds to $X=0$, $Y=0$, ...)

$$\Omega=\Omega_0+[\alpha\alpha]X^2+[\beta\beta]Y^2+\dots+2[\alpha\beta]XY+\dots(5)$$

47. ***Problem.*** To find the weight of the determination $u=u_0$, where u is any linear function of $x, y, z, \dots$ and u_0 its most plausible value.

Suppose $\quad u=lx+my+nz+\dots+h$

then $\quad u_0=lx_0+my_0+nz_0+\dots+h.$

Also, when $x, y, z, \dots$ are expressed (Art. 41) in terms of $X, Y, Z, \dots$ suppose the value of u becomes

$$u=u_0+\mathfrak{A}X+\mathfrak{B}Y+\mathfrak{C}Z+\dots$$

If the system be disturbed by giving u a value different from u_0, we have (f being the force acting on u)

$$fdu=\tfrac{1}{2}d\Omega=Xdx+Ydy+Zdz+\dots$$

and the conditions of relative equilibrium are

$$\frac{X}{l}=\frac{Y}{m}=\frac{Z}{n}=\dots\dots \qquad (A)$$

whence (Art. 18)

$$\frac{\mathfrak{A}X+\mathfrak{B}Y+\mathfrak{C}Z+\ldots}{l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}+\ldots}=\frac{Xdx+Ydy+Zdz+\ldots}{ldx+mdy+ndz}$$

$$=\frac{1}{2}\frac{d\Omega}{du}=f$$

or

$$f=\frac{u-u_0}{l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}+\ldots}$$

which shews that the weight G of the determination $u=u_0$ is given by the equation

$$\frac{1}{G}=l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}+\ldots$$

Another expression for G may be obtained thus.

Since

$$f=\frac{X}{l}=\frac{Y}{m}=\frac{Z}{n}=\ldots.$$

multiply the several terms of the equations in Art. 41 by the reciprocals of these quantities, and there results

$$x-x_0=f.(l\,[\alpha\alpha]+m\,[\alpha\beta]+n\,[\alpha\gamma]+\ldots)$$
$$y-y_0=f.(l[\alpha\beta]+m\,[\beta\beta]+n\,[\beta\gamma]+\ldots)$$
$$z-z_0=f.(l\,[\gamma\alpha]+m\,[\beta\gamma]+n\,[\gamma\gamma]+\ldots)$$

&c. whence, multiplying by l, m, n,... and adding,

$$u-u_0=f.\,\{l^2\,[\alpha\alpha]+m^2\,[\beta\beta]+\ldots.+2lm[\alpha\beta]+\ldots\}$$

or

$$f=G.(u-u_0)$$

where $\frac{1}{G}$ is the expression which would result from writing l, m, n,... instead of X, Y, Z,... in the value of $\Omega-\Omega_0$ ((5) Art. 46). This agrees with the general theorem of Art. 28. See also *Theor. Comb.* Art. 29.

48. Since

$$fdu = \tfrac{1}{2}d\Omega = G\,(u - u_0)\,du,$$

we have by integration

$$\Omega = \Omega_0 + G\,(u - u_0)^2$$

an equation which subsists only in conjunction with the conditions (A). (See Art. 29.)

The equation which determines systems of equal plausibility is as before (Art. 30)

$$\Omega = \Omega_0 + \epsilon^2$$

and if we suppose the arbitrary ϵ^2 to have the greatest value which is to be reasonably feared, the same equation (Art. 31) is the expression of the general want of precision of the result. The error to be feared in the determination of the function u, is $\pm \frac{\epsilon}{\sqrt{G}}$, which becomes in the particular cases of $u = x$, $u = y$, $u = z$, ...

$\pm \epsilon\sqrt{[\alpha\alpha]}$, $\pm \epsilon\sqrt{[\beta\beta]}$, $\pm \epsilon\sqrt{[\gamma\gamma]}$, ... respectively. (Art. 42.) (*Theor. Comb.* Art. 30.)

It is to be remembered, that these extreme errors are not to be apprehended for x, y, z, ... *all at once;* if we suppose for example the error of x to be $\pm \epsilon\sqrt{[\alpha\alpha]}$, we have only to fear for y, z, . . . such combinations of errors as may be consistent with the equation $\Omega = \Omega_0 + \epsilon^2$.

I now proceed to shew how the value of ϵ^2 is to be estimated *à posteriori.*

SECTION VIII.

49. Let σ be the number of observations, or of the functions $v, v', \dots$ and τ the number of unknown quantities $x, y, z, \dots$.

If we take any system of σ coefficients

$$l, l', l'', \dots\dots$$

which satisfy the τ equations

$$\left.\begin{array}{l} al + a'l' + a''l'' + \dots = 0 \\ bl + b'l' + b''l'' + \dots = 0 \end{array}\right\} \quad (1)$$

&c. or $\Sigma(al) = 0, \Sigma(bl) = 0, \dots\dots$ we have identically,

$$lv + l'v' + l''v'' + \dots\dots = lk + l'k' + l''k'' + \dots\dots \quad (2)$$

or

$$\Sigma\,(lv) = \Sigma\,(lk)$$

Now the observations gave $v = 0, v' = 0, \dots\dots$ with weights $= 1$ respectively; so that the value of $\Sigma\,(lv)$ deduced immediately[r] from them is 0, and the weight of this determination would be (Art. 19.) the reciprocal of $l^2 + l'^2 + l''^2 + \dots\dots$ or of $\Sigma(l^2)$. If therefore we assume, for convenience, in addition to the equations (1) the further condition

$$\Sigma\,(l^2) = 1$$

we shall have, by (2) the actual error e of a determination whose weight is unity, viz.

$$e = \Sigma\,(lk)$$

(supposing errors positive which give the quantity to be determined too small).

---

[r] i. e. Leaving out of sight the existence of equations of condition, and treating $v, v', \dots$ as if they were independent.

Inasmuch as the elimination of the τ quantities $x, y, z, \ldots$ between the σ equations

$$v = ax + by + cz + \ldots$$
$$v' = a'x + b'y + c'z + \ldots$$

&c. can only produce $\sigma - \tau$ essentially different equations of condition between $v, v', \ldots.$ it follows that we can in general find $\sigma - \tau$ essentially different equations such as (2), and no more. That is, if $\sigma - \tau = i$, we can find i distinct identical relations between $v, v', v'', \ldots.$ viz.

$$\Sigma(l_1 v) = \Sigma(l_1 k),\ \Sigma(l_2 v) = \Sigma(l_2 k), \ldots. \Sigma(l_i v) = \Sigma(l_i k)$$

where each of the i systems of σ coefficients

$$\begin{array}{l} l_1,\ l'_1,\ l''_1, \ldots. \\ l_2,\ l'_2,\ l''_2 \ldots. \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ l_i,\ l'_i,\ l''_i \ldots. \end{array}$$

is subject to a system of τ equations

$$\Sigma(al) = 0,\ \Sigma(bl) = 0, \ldots\ldots.$$

Moreover it is evident that we may add, not only the i conditions $\Sigma(l_1^2) = 1$, $\Sigma(l_2^2) = 1, \ldots \Sigma(l_i^2) = 1$ which will give unity for the weight of each of the determinations (derived directly from the observations)

$$\Sigma(l_1 v) = 0,\ \Sigma(l_2 v) = 0, \ldots \Sigma(l_i v) = 0$$

but also the $\dfrac{i(i-1)}{2}$ conditions

$$\Sigma(l_1 l_2) = 0,\ \Sigma(l_2 l_3) = 0, \ldots. \Sigma(l_p l_q) = 0$$

which (Art. 36) will make these determinations mutually *independent*. For the whole number of coefficients is σi, and the whole number of conditions assumed will be $i\tau + i + \frac{i(i-1)}{2} = i\left(\sigma - \frac{i-1}{2}\right)$, which is less than σi, except in the extreme case in which $i = 1$, when it becomes *equal* to σi.

If then the number of observations only exceed that of the unknown quantities by unity, there is generally only one way of choosing the coefficients $l, l', \ldots$ so as to fulfil all the conditions supposed. But if it be greater, these quantities may be chosen in an infinite number of different ways, each of which however will, so far as our present purpose is concerned, lead to the same result, as will appear immediately.

50. Upon the whole, we can find the actual errors $e_1 = \Sigma(l_1 k)$, $e_2 = \Sigma(l_2 k) \ldots e_i = \Sigma(l_i k)$ of i independent determinations whose weights are severally unity.

Now since $v, v', v'', \ldots$ are subject to the i equations of condition

$$\Sigma(l_1 v) = \Sigma(l_1 k),\ \Sigma(l_2 v) = \Sigma(l_2 k), \ldots \Sigma(l_i v) = \Sigma(l_i k)$$

and to no others essentially different from these, it follows that the minimum value of Ω or $\Sigma(v^2)$ would be determined by the system of equations

$$\Sigma(v dv) = 0,\ \Sigma(l_1 dv) = 0, \ldots \ldots \Sigma(l_i dv) = 0,$$

or, which is the same thing, by the equations

$$v = P_1 l_1 + P_2 l_2 + \ldots + P_i l_i$$
$$v' = P_1 l'_1 + P_2 l'_2 + \ldots + P_i l'_i$$

&c. where $P_1, P_2, \ldots P_i$ are indeterminate multipliers. If we multiply these equations by $l_1, l'_1, \ldots$ respectively, and then add them, taking account of the conditions assumed in the last article, we get

$$\Sigma(l_1 v) = P_1 \text{ or } P_1 = \Sigma(l_1 k) = e_1$$

similarly, using successively each system of coefficients l as multipliers, we should find

$$P_2 = e_2, \ldots P_i = e_i,$$

and therefore the equations above written become

$$v = e_1 l_1 + e_2 l_2 + \ldots + e_i l_i$$
$$v' = e_1 l'_1 + e_2 l'_2 + \ldots + e_i l'_i$$

&c. and these are the values of v, v', ... which make Ω a minimum. If we now add their squares, we get immediately (by virtue of the conditions) for the minimum in question,

$$\Omega_0 = e_1^2 + e_2^2 + \ldots . + e_i^2$$

51. It appears then that the observations themselves can be made to furnish, and generally in an infinite variety of ways, the actual errors of $\sigma - \tau$ independent determinations of which the weight is unity, and the equation just established shews that the sum of the squares of these errors is *always equal to* Ω_0. The average value of the square of the error of such determinations is therefore

$$\frac{\Omega_0}{\sigma - \tau}$$

judging from these $\sigma - \tau$ *examples;* and as we have *no other* examples, we must be content to take this average as the representative of the quantity ϵ^2 (Art. 48), or of the square of the error to be feared in any determination whose weight is unity. The true value of this quantity would be the average taken from an *infinite* number of instances; in proportion therefore as the number $\sigma - \tau$ is greater, the greater will be the confidence with which the value just found may be taken as the true one; but however small $\sigma-\tau$ may be, i. e. however small the probability of the truth of the equation $(\sigma-\tau)\epsilon^2=\Omega_0$, the value of ϵ^2 derived from it is always the *most plausible* value, since there is nothing to decide whether it ought to be *diminished* or *increased.*

The equation $\Omega=\left(1+\frac{1}{\sigma-\tau}\right)\Omega_0$ is therefore, as was stated by anticipation in Art. 38, the best representation which can be got of the general want of precision of the result. And the error to be feared in any determination $u=u_0$, whose weight is G, is $\pm\sqrt{\frac{\Omega_0}{(\sigma \quad G}}$ (Art. 31).

52. These results correspond in a remarkable way with those furnished by the theory of probabilities. It is proved namely (see *Theor. Comb.* Art. 38) by that theory, that if μ^2 be the true value of the quantity denoted above by ϵ^2, that is to say, the average value of the square of the error of a determination whose weight is unity, deduced from an infinite number of instances; and M the average value which Ω_0 would have on an infinite number of repetitions of the system of observations supposed in the problem, then

$$\mu^2=\frac{M}{\sigma-\tau} \qquad (1)$$

and as M is unknown, the best representative of its value is the value of Ω_0 in the *actual case,* so that the most plausible value of μ^2 is $\frac{\Omega_0}{\sigma-\tau}$, as above.

Thus the theory of probabilities shews that equation (1) is true when μ^2 and M have the average values which would be deduced respectively from an *infinite number of cases.* The investigations of the present section shew that the same equation is true when μ^2 and M have such average values as can be obtained from the *actual case;* which furnishes $\sigma - \tau$ instances for the former, and *one* instance for the latter quantity.

53. I shall conclude this subject with a geometrical illustration, which throws considerable light upon it.

Suppose three direct observations, with weights respectively equal to unity, have given

$$\xi = 0, \ \eta = 0, \ \zeta = 0$$

ξ, η, ζ, being rectangular coordinates of a point in space, whose position is to be determined. Suppose moreover that the point is known to be in a certain straight line passing through a given point and in a given direction. This is equivalent to assuming two equations of condition

$$\left.\begin{aligned} l(\xi - a) + m(\eta - b) + n(\zeta - c) = 0 \\ l'(\xi - a) + m'(\eta - b) + n'(\zeta - c) = 0 \end{aligned}\right\} \ (1)$$

together with the relations

$$\left.\begin{aligned} l \cos\alpha + m\cos\beta + n\cos\gamma = 0 \\ l'\cos\alpha + m'\cos\beta + n'\cos\gamma = 0 \end{aligned}\right\} \ (2)$$

where a, b, c, are the coordinates of the given point, and α, β, γ, the given angles which the line makes with the axes of coordinates. For, the conditions (2) being supposed, the equations (1) obviously represent two planes whose intersection is the line in question.

Now we deduce directly from the observations

$$l\xi + m\eta + n\zeta = 0$$

and

$$l'\xi + m'\eta + n'\zeta = 0$$

the errors of which deductions are by (1)

$$e = la + mb + nc$$
$$e' = l'a + m'b + n'c$$

If we further assume

$$l^2 + m^2 + n^2 = 1, \; l'^2 + m'^2 + n'^2 = 1,$$

then e and e' represent the perpendicular distances of the origin, (the point assigned directly by the observations) from the two planes (1), in each of which the point really is. The same conditions also express that the weight of each of the two deductions above written is unity, (Art. 19.) Let us also assume

$$ll' + mm' + nn' = 0$$

the two deductions will then (Art. 36) be *independent*. This last assumption only expresses that the two planes (1), whose intersection determines the given straight line, are at right angles to one another. All the conditions thus assumed may be satisfied in an infinite variety of ways, since the two planes may still turn round the line of their intersection, remaining always at right angles to one another. But whatever particular position we give them, subject to these conditions, it is obvious that the sum of the squares of the two errors e, e', that is of the distances of the origin from the two planes at right angles to one another, will always be the same, and will be equal to the square of the perpendicular distance of the origin from the given line round which they turn; which is also evidently equal to the least value which $\xi^2+\eta^2+\zeta^2$, or Ω, can have consistently with the given conditions.

The most plausible position of the point is then that point of the given straight line which is nearest to the origin; and the best expression to be got for the *want of precision* of this result is

$$\xi^2+\eta^2+\zeta^2=\Omega_\circ+\epsilon^2$$

where $\Omega_\circ$ is easily found to be equal to
$(b\cos\gamma-c\cos\beta)^2+(c\cos\alpha-a\cos\gamma)^2+(a\cos\beta+b\cos\alpha)^2$
and ϵ^2 is the average value of the $(\text{error})^2$ of a determination whose weight is unity, taken from the *two actual instances* e^2 and e'^2; or $\epsilon^2=\frac{1}{2}\Omega_\circ$. Thus the observations may be said to have determined, not a single point in the given straight line, but a portion of it cut off by the sphere whose equation is

$$\xi^2+\eta^2+\zeta^2=\tfrac{3}{2}\Omega_\circ$$

of which portion at the same time the middle point is the *most* plausible position.

SECTION IX.

54. The form of the functions X, Y, Z, ... (Art. 46) naturally suggests some considerations which, though not likely to be practically useful, are nevertheless interesting in other respects, and tend to illustrate the subject generally. I shall therefore make no apology for introducing them here. Let $x-x_\circ=x'$, ... as before, and let us examine the consequences of assuming

$$\frac{X}{x'}=\frac{Y}{y'}=\frac{Z}{z'}=\ldots\qquad(A)$$

Since each of these fractions would (Art. 18.) be equal to $\dfrac{Xx'+Yy'+Zz'+\ldots}{x'^2+y'^2+z'^2+\ldots}$

or, (Art. 46) to $\dfrac{\Omega-\Omega_0}{r^2}$, if $x'^2+y'^2+z'^2+\ldots=r^2$,

we should have, putting $\dfrac{\Omega-\Omega_0}{r^2}=s$,

$$X=sx',\ Y=sy',\ldots.$$

or, (writing for $X, Y, Z, \ldots$ their values, and putting

$$\frac{x'}{r}=l,\ \frac{y'}{r}=m,\ \frac{z'}{r}=n,\ldots.)$$

$$\left.\begin{array}{l}
\big(\Sigma(a^2)-s\big)l+\Sigma(ab)m+\Sigma(ac)n+\ldots\ =0\\
\Sigma(ab)l+\big(\Sigma(b^2)-s\big)m+\Sigma(bc)n+\ldots\ =0\\
\Sigma(ac)l+\Sigma(bc)m+\big(\Sigma(c^2)-s\big)n+\ldots\ =0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}\quad(1)$$

together with the condition

$$l^2+m^2+n^2+\ldots.=1 \qquad (2)$$

These equations are merely a transformation of the assumed conditions (A).

The elimination of all the quantities $l, m, n, \ldots.$ from equations (1) would give a single equation for determining s, and to each real value of s would correspond a system of real values of $l, m, n, \ldots.$ and the assumptions (A) would then be satisfied by supposing

$$\frac{x'}{l}=\frac{y'}{m}=\frac{z'}{n}=\ldots. \qquad (B)$$

and giving $l, m, n, \ldots.$ any one of the systems of values thus obtained.

55. Let τ be the number of the quantities $x, y, z, \ldots$ or $X, Y, Z, \ldots.$ Then there are τ equations (1), and the result of the elimination of $l, m, n, \ldots$

will be of the degree τ with respect to s; for it will contain, as is easily seen, the product

$$\big(\Sigma(a^2)-s\big)\ \big(\Sigma(b^2)-s\big)\ \big(\Sigma(c^2)-s\big)\ \ldots\ldots$$

(In fact, if we eliminate l between the first equation and each of the rest, we shall get $\tau-1$ equations free from l, and the coefficient of m in the first of them will contain the term $\big(\Sigma(a^2)-s\big)\ \big(\Sigma(b^2)-s\big)$. If we then eliminate m between this first equation and each of the rest, we shall get $\tau-2$ equations free from m, in the first of which the coefficient of n will contain $\big(\Sigma(a^2)-s\big)\ \big(\Sigma(b^2)-s\big)\ \big(\Sigma(c^2)-s\big)$. And so on till we arrive at a single equation free from all the quantities l, m, n, ... and containing the product of all the factors above written.)

Let s_1, s_2, s_3, ... be the τ roots of this equation, (which we will denote by $S=0$), and l_1, m_1, n_1, ... the values of l, m, n, ... corresponding to s_1; l_2, m_2, n_2, ... those corresponding to s_2; &c. We have then from the first of equations (1) in the last article

$$\left.\begin{aligned}\Sigma(a^2)l_1+\Sigma(ab)m_1+\Sigma(ac)n_1+\ \ldots\ &=s_1\,l_1\\ \Sigma(a^2)l_2+\Sigma(ab)m_2+\Sigma(ac)n_2+\ \ldots\ &=s_2\,l_2\end{aligned}\right\}\ (1)$$

with two similar equations from each of the rest.

Eliminating $\Sigma(a^2)$, $\Sigma(b^2)$, $\Sigma(c^2)$ ... respectively from these successive pairs of equations, we get,

$$(l_1m_2-l_2m_1)\ \Sigma(ab)+(l_1n_2-l_2n_1)\ \Sigma(ac)+\ldots\ =(s_2-s_1)\,l_1l_2$$

$$(m_1l_2-m_2l_1)\ \Sigma(ba)+(m_1n_2-m_2n_1)\ \Sigma(bc)+\ldots\ =(s_2-s_1)\,m_1m_2$$

$$(n_1l_2-n_2l_1)\ \Sigma(ca)+(n_1m_2-n_2m_1)\ \Sigma(cb)+\ldots\ =(s_2-s_1)\,n_1n_2$$

&c. If we add these equations, the terms on the left side mutually destroy one another, and the result is

$$(s_2-s_1)\ (l_1l_2+m_1m_2+n_1n_2+\ \ldots.)=0 \qquad (\text{I})$$

and consequently, unless the two roots s_1, s_2 are *equal*,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 + \ldots \quad = 0\text{[a]}$$

56. The equation (I) just proved, leads very easily to the conclusion that all the roots $s_1\ s_2 \ldots$ of the equation $S = 0$ are necessarily real.

For, if possible, let s_1, s_2, be a pair of conjugate roots, and suppose $s_1 = p - q\sqrt{-1}$, $s_2 = p + q\sqrt{-1}$. From the symmetry of the conditions (1) and (2), Art. 54, it follows that the values of l_1, l_2, m_1, m_2, ... will also be respectively conjugate, so that we may assume

$$l_1 = \lambda - \alpha\sqrt{-1},\ m_1 = \mu - \beta\sqrt{-1}, \ldots$$

$$l_2 = \lambda + \alpha\sqrt{-1},\ m_2 = \mu + \beta\sqrt{-1}, \ldots$$

and equation (I) of the last Article becomes

$$2q\sqrt{-1}\,.\,(\lambda^2 + \mu^2 + \ldots + \alpha^2 + \beta^2 + \ldots) = 0$$

Now the last factor cannot vanish unless

$$\lambda = 0,\ \mu = 0, \ldots \alpha = 0,\ \beta = 0, \ldots$$

which would contradict the equations

$$l_1^2 + m_1^2 + \ldots = 1$$

$$l_2^2 + m_2^2 + \ldots = 1$$

and therefore we have necessarily

$$q = 0$$

which proves that if there be a pair of *conjugate* roots, they are not imaginary, but real and *equal;* and consequently there can be no imaginary roots.

57. Since the roots of the equation $S = 0$ are all real, and are moreover particular values of $\frac{\Omega - \Omega_0}{r^2}$, which is essentially positive, they are also necessarily

---

[a] See Cauchy's *Exercices de Mathématiques* for 1828, p. 9.

all positive. (The *reality* of the roots depends only on $\Omega - \Omega_0$ being a homogeneous integer function of x', y', z', ... of the 2nd degree; but they would not necessarily be all positive if $\Omega - \Omega_0$ were not essentially positive.) Let the reciprocals of the roots be denoted by A^2, B^2, C^2, ...

Now let the variables x', y', z', ... be expressed in terms of new independent variables ξ, η, ζ, ... (the same in number) by means of the equations

$$\begin{aligned} x' &= l_1\xi + l_2\eta + l_3\zeta + \dots \\ y' &= m_1\xi + m_2\eta + m_3\zeta + \dots \\ z' &= n_1\xi + n_2\eta + n_3\zeta + \dots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \qquad (\alpha)$$

the relations already established between the coefficients give

$$x'^2 + y'^2 + z'^2 + \dots = \xi^2 + \eta^2 + \zeta^2 + \dots$$

and also

$$\begin{aligned} \xi &= l_1x' + m_1y' + n_1z' + \dots \\ \eta &= l_2x' + m_2y' + n_2z' + \dots \\ \zeta &= l_3x' + m_3y' + n_3z' + \dots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \qquad (\beta)$$

These last equations shew that the values of x', y', z', ... expressed as above, are themselves independent. And since the equations (α) must become identical when the values (β) are substituted for ξ, η, ζ, ... we get the additional relations between the coefficients

$$\begin{array}{ll} l_1^2 + l_2^2 + l_3^2 + \dots \quad = 1 & l_1m_1 + l_2m_2 + l_3m_3 + \dots = 0 \\ m_1^2 + m_2^2 + m_3^2 + \dots = 1 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \end{array}$$

Now equations (α), multiplied respectively by $\Sigma(a^2)$,

$\Sigma(ab)$, $\Sigma(ac)$, ... and added, give by virtue of the conditions (1) &c. of Art. 55,

$$X = s_1 l_1 \xi + s_2 l_2 \eta + s_3 l_3 \zeta + \ldots.$$

and similarly (multiplying by $\Sigma(ab)$, $\Sigma(b^2)$, ... and so on)

$$Y = s_1 m_1 \xi + s_2 m_2 \eta + s_3 m_3 \zeta + \ldots.$$

. . . .

multiply these last equations by x', y', z', ... and add, and the result is

$$Xx' + Yy' + Zz' + \ldots = s_1 \xi^2 + s_2 \eta^2 + s_3 \zeta^2 + \ldots.$$

or (Art. 46, Equation (2))

$$\Omega = \Omega_0 + \frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} + \frac{\zeta^2}{C^2} + \ldots. \qquad (\Omega)$$

Thus the expression of the want of precision of the general result becomes

$$\frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} + \frac{\zeta^2}{C^2} + \ldots \qquad = \epsilon^2,$$

an equation which, if ϵ be arbitrary, also determines the systems of *equal plausibility*.

58. If we give to ξ, η, ζ, ... any values different from their most plausible values ($\xi = 0$, $\eta = 0$, ...) and call f_1, f_2, f_3, ... the forces respectively acting upon these variables, we have (as in Art. 25)

$$\tfrac{1}{2} d\Omega = f_1 d\xi + f_2 d\eta + f_3 d\zeta + \ldots$$

and therefore

$$f_1 = \frac{\xi}{A^2}, f_2 = \frac{\eta}{B^2}, f_3 = \frac{\zeta}{C^2}, \ldots.$$

These equations shew that the *weights* of the determinations $\xi = 0$, $\eta = 0$, $\zeta = 0$, ...

are respectively $\frac{1}{A^2}$, $\frac{1}{B^2}$, $\frac{1}{C^2}$, ... i. e. they are the roots of the equation $S = 0$. The same equations shew that these determinations are mutually *independent;* for since the force on each variable depends only on its own value, its equilibrium is unaffected by any alteration made in the values of the others (see Art. 35).

Thus the original system of σ observations of $v, v', v'', \dots$ is equivalent to a system of τ observations of the independent linear functions $\xi, \eta, \zeta, \dots$ giving their values respectively $= 0$, with the weights stated above. There exist however an infinity of other systems of linear functions possessing similar properties. For let

$$\xi = \lambda_1 \xi' + \lambda_2 \eta' + \lambda_3 \zeta' + \dots$$
$$\eta = \mu_1 \xi' + \mu_2 \eta' + \mu_3 \zeta' + \dots$$

&c. then we shall have

$$\Omega = \Omega_0 + \frac{\xi'^2}{A'^2} + \frac{\eta'^2}{B'^2} + \frac{\zeta'^2}{C'^2} + \dots$$

if the coefficients be assumed so as to fulfil the conditions

$$\frac{\lambda_1 \lambda_2}{A^2} + \frac{\mu_1 \mu_2}{B^2} + \frac{\nu_1 \nu_2}{C^2} + \dots = 0 \qquad (1)$$

. . . .

and if

$$\frac{1}{A'^2} = \frac{\lambda_1^2}{A^2} + \frac{\mu_1^2}{B^2} + \frac{\nu_1^2}{C^2} + \dots \qquad (2)$$

. . . .

and it will appear exactly as in the former case, that the determinations $\xi' = 0$, $\eta' = 0$, $\zeta' = 0, \dots$ are independent, and that their weights are respectively

$$\frac{1}{A'^2},\ \frac{1}{B'^2},\ \frac{1}{C'^2}, \dots$$

Since the determinations $\xi=0$, $\eta=0$, $\zeta=0$, ... are equivalent to the results of independent observations with weights

$$\frac{1}{A^2},\ \frac{1}{B^2},\ \frac{1}{C^2},\ldots$$

it follows (Art. 19) that if $u=\lambda\xi+\mu\eta+\nu\zeta+\ldots$ be any linear function of those quantities, the weight G of the determination of u is the reciprocal of

$$\lambda^2A^2+\mu^2B^2+\nu^2C^2+\ldots.$$

The same expression is also easily derived from the equation $G=\frac{d\Omega}{d\,(u-u_0)^2}$, with the conditions of relative equilibrium $\frac{\xi}{\lambda A^2}=\frac{\eta}{\mu B^2}=\ldots.$ (Art. 29.)

The conditions (1) and (2) lead easily to a set of reciprocal conditions involving as a consequence

$$A'^2+B'^2+C'^2+\ldots=A^2+B^2+C^2+\ldots..$$

The functions ξ, η, ζ, ... may be called the *principal* linear functions; and any system ξ', η', ζ', ... a system of conjugate linear functions.

59. The geometrical interpretation of the propositions of this Section, in the case in which there are only three unknown quantities, is so obvious, that it will be sufficient merely to state it.

Let x, y, z, be considered as the rectangular coordinates of a point P whose position in space is to be determined by the observations. The most plausible position of P is the point whose coordinates x_0 y_0 z_0 are determined by the equations $X=0$, $Y=0$, $Z=0$. And if we remove the origin to this point, we have for the equation to any surface of *equal plausibility*

$$\Sigma(a^2)x'^2 + \Sigma(b^2)y'^2 + \Sigma(c^2)x'^2 + 2\Sigma(bc)y'z' \\ + 2\Sigma(ca)z'x' + 2\Sigma(ab)x'y' = \epsilon^2$$

This equation, if different values be assigned to ϵ, represents a system of similar and concentric ellipsoids, whose principal axes are proportional to the square roots of the reciprocals of the roots of the equation[s]

$$\left(\Sigma(a^2) - s\right)\left(\Sigma(b^2) - s\right)\left(\Sigma(c^2) - s\right) - \left(\Sigma(bc)\right)^2 \times \\ \left(\Sigma(a^2) - s\right) - \left(\Sigma(ca)\right)^2 \left(\Sigma(b^2) - s\right) - \left(\Sigma(ab)\right)^2 \times \\ \left(\Sigma(c^2) - s\right) + 2\,\Sigma(bc)\,.\,\Sigma(ca)\,.\,\Sigma(ab) = 0 \qquad (S)$$

and the three systems of values of l, m, n, determined by the conditions assumed in Art. 54, are the cosines of the angles which these principal axes make respectively with the original axes of coordinates.

If we call ξ, η, ζ, the coordinates of any point referred to these principal axes, the equation to any one of the ellipsoids becomes

$$\frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} + \frac{\zeta^2}{C^2} = \epsilon^2$$

where $\frac{1}{A^2}$, $\frac{1}{B^2}$, $\frac{1}{C^2}$ are the roots of the equation (S). The particular ellipsoid corresponding

---

[t] The reality of the roots of this equation has been already demonstrated indirectly. In this case, in which the equation is only of the third degree, it can be also proved directly that S changes its sign three times between $s = -\infty$ and $s = +\infty$. See Cauchy's paper referred to above.

to $\epsilon^2 = \frac{\Omega_\circ}{\sigma - 3}$ (where σ is the number of observations) represents the want of precision of the result, since we have to apprehend that the point P may be anywhere within it.

If the point be placed anywhere on the surface of one of the ellipsoids of equal plausibility, the force of restitution is in the direction of the normal, and proportional to the distance of the tangent plane from the centre. Or, which is the same thing, if the point be displaced along any diameter, the force of restitution is perpendicular to the conjugate diametral plane, and proportional to its distance from that plane: and conversely, if it be displaced by being forced to be in any given plane not passing through the centre, its position of equilibrium is the intersection of that plane with the diameter conjugate to the parallel diametral plane. If A be the greatest of the quantities A, B, C, and C the least, the direction of the axis of ζ may be called the direction of *greatest precision*, and that of ξ the direction of *least precision*. For, since

$$\frac{l^2 + m^2 + n^2}{l^2 A^2 + m^2 B^2 + n^2 C^2}$$

is necessarily intermediate between the greatest and least of the quantities $\frac{1}{A^2}$, $\frac{1}{B^2}$, $\frac{1}{C^2}$, it follows that the weight of the conclusion $\zeta = 0$ is greater, and that of $\xi = 0$ less, than the weight of any such conclusion as $l\xi + m\eta + n\zeta = 0$, with the condition $l^2 + m^2 + n^2 = 1$: that is, it is more certain that the point is in the principal diametral plane of $\xi\eta$, and less certain that it is in that of $\eta\zeta$, than that it is in any other diametral plane whatever. And generally, the want of precision in any direction is measured by the distance from the origin to the plane which is perpendicular to that direction, and touches the ellipsoid corresponding to $\epsilon^2 = \frac{\Omega_0}{\sigma - 3}$.

The original observations giving $v = 0, v' = 0, \ldots\ldots$ with weight = unity, are equivalent to observations of the distances $\delta, \delta', \ldots$ of the point P from the planes whose equations are $ax + by + cz = k, a'x + b'y + c'z = k', \ldots\ldots$ giving $\delta = 0, \delta' = 0, \ldots\ldots$ with weights expressed by $a^2 + b^2 + c^2, a'^2 + b'^2 + c'^2, \ldots\ldots$ respectively, (since the distance δ of any point (x, y, z) from the plane $v = 0$ is $\frac{v}{\sqrt{a^2 + b^2 + c^2}}$). It appears from what precedes, that this whole system of observations may be considered equivalent to *three* direct observations of the distance of the point from three planes at right angles to one another; the principal diametral planes, namely, of the system of ellipsoids determined as above, and that the weights of these supposed observations are inversely proportional to the squares of the principal axes.

60. The general equations $X = 0$, $Y = 0$, $Z = 0$ might, in this case of three unknown quantities only, have been deduced directly as follows.

The observation which gives $v = 0$ with weight 1, gives 0 for the distance δ of the point P from the plane $(ax + by + cz = k)$, with weight $= a^2 + b^2 + c^2$. Consequently, when P is placed at any point (x, y, z) out of the plane, it is acted on (Art. 9) by a force $= (a^2 + b^2 + c^2)\,\delta$, or $v\sqrt{a^2 + b^2 + c^2}$, urging it towards the plane; and the components of this force parallel to the three axes of coordinates are av, bv, cv, since a perpendicular to the plane makes angles with the axes whose cosines are a, b, c, divided respectively by $\sqrt{a^2+b^2+c^2}$. Each of the other observations gives rise to a force which may be similarly resolved; and thus we have for the conditions of equilibrium $\Sigma\,(av)=0$, $\Sigma\,(bv)=0$, $\Sigma\,(cv)=0$, or $X=0$, $Y=0$, $Z=0$, as before.

APPENDIX.

A.

All direct demonstrations of the parallelogram of forces, all those, namely, which are founded strictly on the consideration of forces applied to a single point, require the assumption of the principle of homogeneity in some form; and this principle itself may be regarded as a particular case of a more general law which may be expressed as follows: "All fundamental laws of nature are strictly *intelligible*." If this assumption be allowed, it follows at once, for instance, that the resultant of two equal forces acting on a point bisects the angle between them; because, if not, its position could not be intelligibly explained: or again, that if θ be the angle between two forces, each equal to P, and R their resultant, then the relation between these quantities must be of the form $R=Pf(\theta)$, because if the equation were not homogeneous with respect to R and P, it would be unmeaning unless the unit of force were given, and this unit could not be intelligibly defined. (The equation need not be homogeneous with respect to θ, because angular magnitude has a natural unit which *can* be intelligibly defined.) It is plain, however, that the legitimacy of this assumption (of strict intelligibility) can never be established by mathematical reasoning, but must rest, if it is to be established at all, upon a metaphysical proof. Whether it be or be not possible to find such a proof in fact, is a question on which it is sufficient to say here that it is unphilosophical to assume beforehand, that is, before the examination of any particular system, either the affirmative or negative of it.

The same may be said of the principle of *continuity* alluded to p. 7, note [b], which must be assumed in order to establish *directly* the theory of parallel forces. In this case, the assumption is not only that the law must be strictly intelligible, but also that the expression of it shall contain no *arbitrary* number. If R be the resultant of two parallel forces each equal to P, and a the distance between them, then we know that $R=2P$ when $a=0$, and if we supposed $R=2nP$ when a is not $=0$, n being any constant number, such a law would be perfectly intelligible; the only objection to it would be that the number n must be essentially *arbitrary*, that is, it would be a number not determined by anything in the conditions of the problem. Of course it is impossible to justify such an assumption as this (the exclusion of arbitrary numbers) by *mathematical* reasoning. It may be observed also that it would be refuted by such instances as the occurrence of the number 2 in the law of gravitation, unless it could be shewn either that such a law is not fundamental in the same sense as the laws of force, or, which is probably true, that the number in question is not arbitrary, but is determined by something in the conditions of the case. (On the subject of homogeneity, see also some remarks appended to Professor Powell's paper on Parallel Lines, printed for the Ashmolean Society, 1842.)

B.

The preceding essay does not, as it is hardly necessary to say, profess to be either a complete or a practical exposition of the subject, and therefore everything has been omitted which was not more or less directly connected with the particular mode of treating it adopted in these pages. Nothing, for instance, has been said on the subject of *constant errors*. In fact, since it is the business of the observer to discover and correct them, we may properly assume that the observations are free from all but accidental errors. Again, in Professor Gauss's treatise so often referred to, the three integrals $\int x\phi(x)\,dx$, $\int x^2\phi(x)\,dx$, $\int x^4\phi(x)\,dx$, are employed; the limits of the integration being $-\infty$ and $+\infty$ in each, and $\phi(x)\,dx$ being the expression of the probability that the error of an observation lies between x and $x+dx$. The first of these integrals is the *constant error*, and is assumed $=0$. The second is the average value of the square of the error, and its reciprocal is assumed to measure the *weight* of the observation. The third is the average value of the fourth power of the error; and certain propositions are established by means of it, which have nothing corresponding to them in the present essay; an omission, however, of which the importance is diminished by the circumstance that the value of the integral in question is always unknown, though it may be estimated approximately from the observations themselves.

Lastly, all investigations depending upon the theory of probabilities, or upon the assumption of a particular form for the function $\phi(x)$, (such as for instance the estimate of what is called the *probable error*) were of course excluded by the nature of the case.

The reader is requested to make the following corrections.

P. 5, line 6, for x read x_1

P. 28, note [1], line 1, read $X=\left(\frac{dX}{dx}\right)_0(x-x_0)+\ldots\ldots$

P. 28, note [1], line 2, dele $\left(\frac{dX}{dx}\right)_0=\frac{X}{x-x_0}$

P. 30, line 5 from below, for $\frac{G}{E}$ read $\frac{E}{G}$

P. 43. line 6, for $ny+mz$ read $ny-mz$.